UNIVERSITEIT GENT

Harmonic and Clifford analysis in superspace

Harmonische analyse en cliffordanalyse in superruimten

Hendrik De Bie

Promotor: Prof. Dr. Franciscus Sommen

Proefschrift voorgelegd aan de Faculteit Wetenschappen van de Universiteit Gent tot het behalen van de graad van Doctor in de Wetenschappen: Wiskunde

Universiteit Gent — Faculteit Ingenieurswetenschappen
Vakgroep Wiskundige Analyse — Onderzoekseenheid Cliffordanalyse
Academiejaar 2007–2008

In old days books were written by men of letters and read by the public.
Nowadays books are written by the public and read by nobody.
O. Wilde, A few Maxims for the Instruction of the Over-Educated.

Woord vooraf

In dit woord vooraf zou ik graag een aantal mensen willen bedanken die zowel op wetenschappelijk als op persoonlijk gebied bijgedragen hebben tot de succesvolle voltooiing van mijn doctoraatsonderzoek.

In de eerste plaats uiteraard mijn promotor Frank Sommen. Zijn inzicht in het formuleren en uitbouwen van werkbare wiskundige theorieën is werkelijk opmerkelijk en ik ben hem dankbaar voor alle wijsheid die hij met mij gedeeld heeft. Zijn hoogstpersoonlijke wiskundige stijl heeft mij erg beïnvloed en zal ook mijn toekomstig werk blijven beïnvloeden.

Ook Fred Brackx wil ik graag bedanken. Niet alleen voor het nalezen van al mijn manuscripten en het formuleren van kritische opmerkingen, maar misschien nog meer voor het gestelde vertrouwen.

In het bijzonder wat betreft onderhavig proefschrift wil ik verder David Eelbode en Kevin Coulembier bedanken voor het nalezen en de vele opmerkingen.

Ook de andere (huidige en voormalige) leden van de onderzoekseenheid cliffordanalyse wil ik graag bedanken voor de voorbije jaren. Zij zorgden voor een aangename en vriendschappelijke sfeer, en waren goed gezelschap op vele congressen in binnen- en buitenland.

Hans, Samuel en Wendy van het secretariaat wil ik graag bedanken voor alle technische steun, de zeer vlotte hulp bij tal van praktische zaken en niet in het minst voor de goede verstandhouding.

Verder wil ik ook de andere collega's van de vakgroep Wiskundige Analyse en uiteraard ook van de vakgroep Zuivere Wiskunde en Computeralgebra bedanken voor de aangename sfeer. In het bijzonder de vele avonden na het werk in 'Café De Pinte' zullen me nog lang bijblijven.

Mijn vrienden wil ik bedanken om te luisteren naar al mijn jeremiades als het eens wat minder ging en om me te confronteren met het feit dat er nog een andere, echte wereld buiten de universiteit is.

Ook mijn ouders wil ik graag bedanken. Zij hebben me altijd gestimuleerd om intellectuele keuzes in mijn leven te maken, en dat apprecieer ik enorm.

Tot slot ben ik ook het Fonds voor Wetenschappelijk Onderzoek - Vlaanderen erkentelijk voor het aspirantenmandaat dat me in staat heeft gesteld dit onderzoek te voltooien.

Gent, 2 oktober 2008.

Table of Contents

Chapter 1

Introduction

Examinations, sir, are pure humbug from beginning to end.
If a man is a gentleman, he knows quite enough, and if
he is not a gentleman, whatever he knows is bad for him.
O. Wilde, The Picture of Dorian Gray.

In this chapter we first give an introduction to Clifford analysis, which is a hypercomplex function theory of functions taking values in a Clifford algebra. This introduction is very brief, as all results in this thesis will contain the classical theory as a special case. Next we will explain what a so-called superspace is and sketch some of the different approaches to this field of mathematics.

Finally we will discuss the main aim of the present work, namely the construction of a theory of harmonic and Clifford analysis in these superspaces. We will give a short overview of the results obtained and of the structure of the following chapters.

1.1 Clifford analysis: a hypercomplex function theory

One of the main aims of Clifford analysis is to construct a first order operator, the so-called Dirac operator, factorizing the Laplace operator and to study the function-theoretical properties of the null-solutions of this operator. When working over $\mathbb{R}^m$, this Dirac operator is defined by

$$\partial_{\underline{x}} = \sum_{i=1}^{m} e_i \partial_{x_i},$$

where the e_i form an orthonormal basis of $\mathbb{R}^m$ and satisfy the defining relations of the Clifford algebra $\mathbb{R}_{0,m}$

$$e_i e_j + e_j e_i = -2\delta_{ij}.$$

It is then easily checked that indeed $\partial_{\underline{x}}^2 = \Delta_b$ with

$$\Delta_b = -\sum_{i=1}^{m} \partial_{x_i}^2$$

the Laplace operator in $\mathbb{R}^m$.

Basic references for Clifford analysis are a.o. [13, 38, 58]. For a historical overview we refer the reader to [37]. Detailed accounts of the theory of Clifford algebras can be found in a.o. [19, 80, 74].

When studying the Dirac operator in this setting, we consider functions f which are e.g. elements of spaces such as $C^\infty(\Omega) \otimes \mathbb{R}_{0,m}$, with Ω some open domain in $\mathbb{R}^m$. This means that f can be written as

$$f = \sum_{A} f_A(x_i) e_A$$

with $f_A(x_i) \in C^\infty(\Omega)$ and $e_A = e_{j_1} \ldots e_{j_k}$, $1 \leq j_1 < \ldots < j_k \leq m$.

A function f is then called (left) monogenic if $\partial_{\underline{x}} f = 0$. As the Dirac operator is a generalization of the Cauchy-Riemann operator in two-dimensional complex analysis (factorizing the two-dimensional Laplace operator), monogenic functions have several properties similar to holomorphic functions in the complex plane. To name a few (see e.g. [13]), we have a Cauchy formula, Morera's theorem, Liouville's theorem, Taylor and Laurent series developments, etc. More details will be given later, where necessary.

In recent years, the theory of Clifford analysis has been developed in several extensions of the framework sketched above, such as hermitean Clifford analysis (see [90, 9, 10]), Clifford analysis on manifolds (see [21, 70, 71]), other special first order systems (see [91, 22]), higher spin equations such as Rarita-Schwinger systems (see [16, 17]), etc.

Moreover, the theory of Clifford analysis has been applied in other branches of mathematics, physics and engineering, such as in wavelet theory (see [14, 12, 35]), numerical boundary value problems (see [60]), Maxwell theory (see [68, 72]), etc.

1.2 Superspaces and supermanifolds

Superspaces were developed during the second half of the previous century by Berezin (see [7, 8]) in order to describe bosonic and fermionic fields on equal footing. To that end, one needs to consider functions which are not only functions of commuting variables x_i but also of a set of anti-commuting variables $\grave{x}_j$ satisfying $\grave{x}_j\grave{x}_k = -\grave{x}_k\grave{x}_j$. These superspaces and corresponding supermanifolds now play an important role in theoretical physics, in subjects ranging from superstringtheory over supergravity to random-matrixtheory.

As to the mathematical study of superspaces, there are several different approaches known in the literature. Without claiming completeness, we mention some of them. First of all, we have the approach from modern algebraic geometry (as advocated by Berezin and others, see [8, 73, 69]), where a supermanifold is defined as a differentiable manifold where the structure sheaf is replaced by a sheaf of $\mathbb{Z}_2$-graded algebras satisfying certain technical requirements. Another important approach with perhaps a more physical flavour is based on differential geometry (see a.o. [47, 82, 85]). In this case, a supermanifold is obtained by glueing together flat superspaces of the form $\mathbb{R}^{p|q} = (\Lambda_0)^p \times (\Lambda_1)^q$ where Λ_0 (Λ_1) is the even (odd) part of some finite or infinite-dimensional Grassmann algebra Λ. In other words, the commuting and anti-commuting variables are really interpreted as variables instead of as formal objects in the sense of Berezin. Furthermore, one can prove that these two approaches to supermanifolds are equivalent in a categorical sense (see [5]).

There exist also several other approaches to superspace, where we mention a.o. [103, 67, 66]. For an overview we refer the reader to the book [4].

1.3 The aim of this thesis

The main aim of this thesis is to study superspaces (where we focus on the flat case) from yet another point of view, namely using methods from harmonic and Clifford analysis. This means that we have to construct a set of operators such as a Laplace and a Dirac operator, acting on functions depending on both commuting and anti-commuting variables. We will show that, using proper definitions of such operators, several results of harmonic and Clifford analysis can be transferred to this new framework. First of all, it is possible to build a complete theory of spherical harmonics in superspace, and accordingly one can to some extent introduce orthogonal polynomials in superspace. Using these spherical harmonics, we can introduce an integral over the supersphere and an integral over the whole superspace in a unique way. This is the main result of this thesis, as we will prove that the resulting integral is equivalent with the so-called Berezin integral (see [8]). This means that the ad hoc-notion of this integral can be given a mathematically sound definition based on harmonic analysis. Furthermore we will study the fundamental solution of the super Dirac and Laplace operators and all their natural powers. We will also study several integral transforms in superspace, such as a Fourier transform, a fractional Fourier transform and a Radon transform. Finally, we will also prove a Cauchy integral formula in superspace, which will enable us to draw the analogy with classical complex analysis.

When working on this thesis, we have published several papers on these topics, namely [29, 32, 30, 28, 31, 23, 24]. We have also submitted several preprints for publication, see [27, 26, 25]. Not all published results are reproduced here, as we wanted to present the topic in a uniform and coherent way. The interested reader may hence wish to consult these papers for more information on certain topics. Furthermore, we hope that this work might be interesting to both the Clifford analysis community and researchers working in superspaces. For the Clifford analysis researchers this thesis will provide extensions of several results they are familiar with, but the proofs are mostly done in a different and more general way. For the researchers in supertheories we hope that this thesis contains an interesting framework which might help them to formulate their problems in a more unified way, and perhaps even help solving them.

Let us finally discuss the content of the next chapters in some more detail. In chapter 2 we will introduce the basic superspace framework needed for our purposes. We will introduce sets of orthogonal and symplectic Clifford algebra generators, that will be paired with the commuting and anti-commuting variables. Next we will introduce the basic differential operators needed in the sequel: the Dirac operator, the Laplace operator, the Euler operator and the Laplace-Beltrami operator. We will also study the commutation relations that exist between these operators. As a consequence we will obtain a representation of the Lie algebra $\mathfrak{sl}_2(\mathbb{R})$ and the Lie superalgebra $\mathfrak{osp}(1|2)$ in superspace, which means that we have indeed built a theory of harmonic and Clifford analysis in superspace.

In chapter 3 we will give a detailed study of spherical harmonics and spherical monogenics in superspace, defined as polynomial null-solutions of the Laplace, resp. Dirac operator in superspace. After some technical lemmata containing calculational properties of these operators, we will first study the Fischer decomposition. This is the direct-sum decomposition of the polynomial algebra generated by the commuting and anti-commuting variables into products of spaces of spherical harmonics with powers of a generalized norm squared of a vector. We will determine projection operators selecting the different components in this decomposition in two different ways and we will also give some graphical interpretations. Next, we will give a detailed description of spaces of spherical harmonics: we will decompose such spaces into irreducible pieces under the action of $SO(m) \times Sp(2n)$, which is the natural group action to be considered. Finally, we will refine the results to the case of spherical monogenics. We will also obtain bases of spaces of spherical monogenics using a Cauchy-Kowalewskaia extension technique.

In chapter 4 we will introduce two sets of polynomials in superspace: the Clifford-Hermite and the Clifford-Gegenbauer polynomials. We will exploit an analogy with the standard Euclidean case to obtain good definitions for them in superspace and we will prove several interesting properties such as recursion formulae, differential equations, Rodrigues formulae and relations with classical orthogonal polynomials on the real line. We will also show that the Clifford-Hermite functions, which are the Clifford-Hermite polynomials multiplied with a suitable Gaussian function, are solutions of the harmonic oscillator in superspace. This will give us an interpretation of the super-dimension, a numerical parameter introduced in chapter 2 to describe general properties of superspaces.

In chapter 5 we will discuss integration in superspace. We will begin with integration over the supersphere. We will define an integral over the supersphere as a linear functional mapping the space of polynomials to the real numbers, invariant under the action of $SO(m) \times Sp(2n)$ and invariant with respect to the algebraic equation of the supersphere. Using the results of chapter 3 we will prove that the set of functionals satisfying these properties is a finite-dimensional vectorspace. If moreover we demand that spaces of spherical harmonics of different degree are orthogonal with respect to integration over the supersphere, there remains, up to a constant, only one functional satisfying these properties. We will call this functional the Pizzetti integral. In the following sections we will investigate the properties of this integral and prove a.o. a Funk-Hecke theorem in superspace. In the final section we will extend the Pizzetti integral to the whole superspace using a generalized integration in spherical co-ordinates. We will prove that the result obtained is equivalent with the so-called Berezin integral.

In chapter 6 we will use the previously obtained integral to define a delta distribution in superspace, allowing us to formulate the question of a fundamental solution for the Dirac and Laplace operator. We will then develop a method to construct this fundamental solution for all natural powers of the Dirac operator.

In chapter 7 we will develop the theory of the Fourier transform in superspace. We will first make a detailed study of the purely fermionic Fourier transform, where the kernel is the basic symplectic invariant (contrary to the orthogonal kernel used by other authors). We will obtain an explicit description of the action of this Fourier transform on the Grassmann algebra. Next we will combine this transform with the classical Fourier transform to obtain a general transform in superspace. We will prove the basic properties of this transform such as inversion, Parseval theorem, etc. We will also prove that the Clifford-Hermite functions defined in chapter 4 diagonalize the Fourier transform. This will allow us to obtain an operator-exponential expression for the transform and also to define a so-called fractional Fourier transform. We will show how one can obtain an integral representation of this new transform. Finally we will extend the Radon transform to superspace by exploiting the central-slice theorem that gives a connection between the classical Radon transform and the classical Fourier transform. It will turn out that the Clifford-Hermite functions will again transform nicely under this generalized Radon transform.

In chapter 8 we will establish a Cauchy integral formula in superspace, thus making the analogy with classical complex analysis complete. We will first state the bosonic Cauchy theorem, known from Clifford analysis. Then we will exploit a nice analogy with the Grassmann algebra to obtain a purely fermionic Cauchy theorem (by making use of some rather complicated computations). In order to obtain a general Cauchy theorem we then need to define in a suitable way integration over the boundary of a manifold. It will turn out that this boundary will consist of the normal (even) boundary and a new part, which is the odd boundary. Using these definitions it is possible to obtain a general Cauchy theorem in superspace, which formally looks completely the same as the usual Cauchy theorem in Clifford analysis. Finally we will obtain some consequences of this theorem. We will a.o. prove a Cauchy-Pompeiu representation formula for monogenic functions, using the fundamental solution of the Dirac operator determined in chapter 6.

In chapter 9 we will first present some conclusions regarding our work. Then we will reflect on some possible directions for further research. First of all, it would be interesting to obtain compatibility conditions for solving systems of Laplace or Dirac operators in superspace. Secondly, we would like to obtain a generalized Mehler formula, connecting the Clifford-Hermite functions with the kernel of the fractional Fourier transform. Another important topic is the development of a quantum mechanics in superspace and more specifically the study of Schrödinger operators. Finally we would like to extend the theory of Dunkl operators to superspace and see to what extent this changes the development of the theory.

Chapter 2

Clifford analysis in superspace

In this chapter, we introduce the framework necessary to build a theory of Clifford and harmonic analysis in superspace. In the first section we introduce a superspace with m commuting and $2n$ anti-commuting variables. The commuting variables will be linked with an orthogonal Clifford algebra, whereas the anti-commuting ones will be linked with a symplectic Clifford algebra. We will introduce the function spaces that will be used in the sequel. Next we will define a set of differential operators that will be of the utmost importance for developing our theory, such as a Dirac and Laplace operator, an Euler operator, a Laplace-Beltrami operator, a generalization of the norm squared of a vector etc. Finally, we will give the most important commutation relations that exist between these operators. We will also prove that we have a representation of $\mathfrak{sl}_2(\mathbb{R})$ resp. $\mathfrak{osp}(1|2)$ in superspace, which allows us to speak of harmonic resp. Clifford analysis in superspace.

The results of this chapter have been published in [29, 28].

2.1 The superspace framework

In [97, 98, 99] Sommen formulated the basic definitions needed to develop Clifford analysis in superspace. This was based on a framework called radial algebra (see [96]) which was introduced to describe Clifford analysis in a more abstract setting. In this chapter we will work differently. We will first introduce the framework and afterwards prove that the relevant operators satisfy the relations defining $\mathfrak{osp}(1|2)$. We refer the reader interested in radial algebra and related issues to the papers cited above or to [29] where a brief review is given.

We start by introducing the algebra of superpolynomials

$$\mathcal{P} = \mathbb{R}[x_1, \ldots, x_m] \otimes \Lambda_{2n}$$

generated by m commuting variables x_i and $2n$ anti-commuting variables $x\grave{}_i$, subject to

$$\begin{array}{rcll} x_i x_j & = & x_j x_i, & i, j \in 1, \ldots, m \\ x\grave{}_i x\grave{}_j & = & -x\grave{}_j x\grave{}_i, & i, j \in 1, \ldots, 2n \\ x_i x\grave{}_j & = & x\grave{}_j x_i, & i \in 1, \ldots, m; \quad j \in 1, \ldots, 2n. \end{array}$$

The anti-commuting variables $x\grave{}_i$ generate a so-called Grassmann algebra Λ_{2n}.

We will now combine the commuting variables with m orthogonal Clifford algebra generators e_i and the anti-commuting variables with $2n$ symplectic Clifford algebra generators $e\grave{}_i$, thus obtaining objects that behave in such a way that there is no longer a real difference between the commuting and anti-commuting variables. These generators satisfy the following relations:

$$\begin{array}{l} e_j e_k + e_k e_j = -2\delta_{jk} \\[1ex] e\grave{}_{2j} e\grave{}_{2k} - e\grave{}_{2k} e\grave{}_{2j} = 0 \\ e\grave{}_{2j-1} e\grave{}_{2k-1} - e\grave{}_{2k-1} e\grave{}_{2j-1} = 0 \\ e\grave{}_{2j-1} e\grave{}_{2k} - e\grave{}_{2k} e\grave{}_{2j-1} = \delta_{jk} \\[1ex] e_j e\grave{}_k + e\grave{}_k e_j = 0. \end{array}$$

The elements e_i generate a standard orthogonal Clifford algebra $\mathbb{R}_{0,m}$ of signature $(-1, \ldots, -1)$. The elements $e\grave{}_i$ generate a symplectic Clifford algebra or Weyl algebra $\mathcal{W}_n$. The algebra generated by both the e_i and the $e\grave{}_i$ will be denoted by $\mathcal{C}$. We furthermore assume that the variables x_i and $x\grave{}_j$ commute with the Clifford algebra generators e_i and $e\grave{}_j$.

We call $\underline{x} = \sum_{i=1}^{m} x_i e_i$ the bosonic vector variable and $\underline{x\grave{}} = \sum_{j=1}^{2n} x\grave{}_j e\grave{}_j$ the fermionic vector variable. A general vector variable in superspace can then be defined by

$$x = \underline{x} + \underline{x\grave{}}.$$

The square of x is given by

$$x^2 = \sum_{j=1}^{n} x\grave{}_{2j-1} x\grave{}_{2j} - \sum_{j=1}^{m} x_j^2 \tag{2.1}$$

and is a generalization of the norm squared of a vector in Euclidean space.

If we consider a second independent vector variable y, the anti-commutator between x and y is given by

$$\{x, y\} = -2\sum_{j=1}^{m} x_j y_j + \sum_{j=1}^{n} (\grave{x}_{2j-1}\grave{y}_{2j} - \grave{x}_{2j}\grave{y}_{2j-1}). \tag{2.2}$$

Indeed, we clearly have the following relations:

$$\begin{array}{l}
\underline{x}\underline{y} + \underline{y}\underline{x} = -2\sum_{j=1}^{m} x_j y_j \\
\underline{x}\underline{\grave{y}} + \underline{\grave{y}}\underline{x} = 0 = \underline{\grave{x}}\underline{y} + \underline{y}\underline{\grave{x}} \\
\underline{\grave{x}}\underline{\grave{y}} + \underline{\grave{y}}\underline{\grave{x}} = \sum_{j,k} \grave{x}_j \grave{y}_k [\grave{e}_j, \grave{e}_k] = \sum_{j=1}^{n} (\grave{x}_{2j-1}\grave{y}_{2j} - \grave{x}_{2j}\grave{y}_{2j-1})
\end{array}$$

from which the statement follows. In the sequel, we will use formula (2.2) as a generalized inner product, allowing us to define e.g. the Fourier transform in superspace (see chapter 7).

Let us finally introduce some more general function spaces, given by

$$\mathcal{F}(\Omega)_{m|2n} = \mathcal{F}(\Omega) \otimes \Lambda_{2n} \otimes \mathcal{C}$$

where $\mathcal{F}(\Omega)$ stands for $\mathcal{D}(\Omega)$, $C^k(\Omega)$, $L_p(\Omega)$, $L_1^{\mathrm{loc}}(\Omega)$, $\ldots$ with Ω an open domain in $\mathbb{R}^m$.

2.2 Differential operators

We begin by introducing the basic derivatives with respect to the (anti-) commuting variables. The (classical) derivative ∂_{x_j} satisfies

$$\begin{array}{l}
\partial_{x_j}(x_k F) = \delta_{jk} F + x_k \partial_{x_j} F \\
\partial_{x_j}(\grave{x}_k F) = \grave{x}_k \partial_{x_j} F
\end{array}$$

with F some element in $C^1(\Omega)_{m|2n}$. The partial derivative $\partial_{\grave{x}_j}$ with respect to an anti-commuting variable satisfies

$$\begin{array}{l}
\partial_{\grave{x}_j}(\grave{x}_k F) = \delta_{jk} F - \grave{x}_k \partial_{\grave{x}_j} F \\
\partial_{\grave{x}_j}(x_k F) = x_k \partial_{\grave{x}_j} F
\end{array}$$

with F some element in $C^1(\Omega)_{m|2n}$. From this we immediately obtain

$$\begin{array}{l}
\partial_{x_j}\partial_{\grave{x}_k} = \partial_{\grave{x}_k}\partial_{x_j} \\
\partial_{\grave{x}_j}\partial_{\grave{x}_k} = -\partial_{\grave{x}_k}\partial_{\grave{x}_j}
\end{array}$$

so the anti-commuting derivatives $\partial_{x̀_j}$ also generate a Grassmann algebra.

Now we introduce the bosonic Dirac operator $\partial_{\underline{x}}$ and the fermionic Dirac operator $\partial_{\underline{x̀}}$ by

$$\begin{aligned}
\partial_{\underline{x}} &= \textstyle\sum_{j=1}^{m} e_j \partial_{x_j} \\
\partial_{\underline{x̀}} &= 2 \textstyle\sum_{j=1}^{n} \left(è_{2j} \partial_{x̀_{2j-1}} - è_{2j-1} \partial_{x̀_{2j}} \right).
\end{aligned}$$

The left and right (super) Dirac operators $\partial_x \cdot, \cdot \partial_x$ are then defined by

$$\begin{aligned}
\partial_x \cdot &= \partial_{\underline{x̀}} \cdot - \partial_{\underline{x}} \cdot; \qquad & F \rightarrow \partial_{\underline{x̀}} F - \partial_{\underline{x}} F = \partial_x F \\
\cdot \partial_x &= - \cdot \partial_{\underline{x̀}} - \cdot \partial_{\underline{x}}; \qquad & F \rightarrow -F \partial_{\underline{x̀}} - F \partial_{\underline{x}} = F \partial_x
\end{aligned}$$

where one has to mind the extra minus sign in the action from the right. Except for chapter 8 where we will establish a Cauchy formula and for the next calculation, we will always use the left Dirac operator.

If we let the Dirac operator act on the vector variable x we obtain

$$\begin{aligned}
\partial_x x &= (\partial_{\underline{x̀}} - \partial_{\underline{x}}) \left(\sum_{j=1}^{n} (è_{2j-1} x̀_{2j-1} + è_{2j} x̀_{2j}) + \sum_{j=1}^{m} e_j x_j \right) \\
&= 2 \sum_{j=1}^{n} (è_{2j} è_{2j-1} - è_{2j-1} è_{2j}) - \sum_{j=1}^{m} e_j e_j \\
&= m - 2n \\
&= (\underline{x̀} + \underline{x}) (-\partial_{\underline{x̀}} - \partial_{\underline{x}}) \\
&= (x \partial_x)
\end{aligned}$$

where $(x\partial_x)$ means the action of ∂_x on x from the right. Now we define the numerical parameter M by

$$M = m - 2n. \tag{2.3}$$

This parameter will be called the super-dimension. It will appear in several formulae in the next chapters and plays the role of a generalized dimension. However, note that this parameter can be negative.

We can define a super Laplace operator as the square of the Dirac operator. We obtain

$$\Delta = \partial_x^2 = - \sum_{j=1}^{m} \partial_{x_j}^2 + 4 \sum_{j=1}^{n} \partial_{x̀_{2j-1}} \partial_{x̀_{2j}}. \tag{2.4}$$

We denote by $\Delta_b = -\sum_{j=1}^{m} \partial_{x_j}^2$ the bosonic Laplace operator and by $\Delta_f = 4 \sum_{j=1}^{n} \partial_{x̀_{2j-1}} \partial_{x̀_{2j}}$ the fermionic Laplace operator. It is easy to check that $\Delta(x^2) = 2M$.

Remark 1. *In chapter 3 we will show that the fermionic Laplace operator is invariant under the symplectic group. Similarly, the fermionic Dirac operator can be seen as a symplectic Dirac operator. Note that symplectic Dirac operators have also been studied in the framework of symplectic manifolds, we refer the reader e.g. to the book [61].*

Next, we define the bosonic and fermionic Euler operators by means of

$$\begin{aligned}\mathbb{E}_b &= \textstyle\sum_{j=1}^{m} x_j \partial_{x_j}\\ \mathbb{E}_f &= \textstyle\sum_{j=1}^{2n} x\grave{}_j \partial_{x\grave{}_j}\end{aligned}$$

and the super Euler operator as

$$\mathbb{E} = \mathbb{E}_b + \mathbb{E}_f.$$

If we consider a monomial $\phi = x_1^{\alpha_1} \dots x_m^{\alpha_m} x\grave{}_1^{\beta_1} \dots x\grave{}_{2n}^{\beta_{2n}} \in \mathcal{P}$ with $\alpha_i \in \mathbb{N}$ and $\beta_i \in \{0, 1\}$, we immediately obtain

$$\mathbb{E}\phi = k\phi$$

with $k = \sum_{i=1}^{m} \alpha_i + \sum_{i=1}^{2n} \beta_i$. So it still makes sense to speak of homogeneous polynomials of degree k and we obtain the following decomposition

$$\mathcal{P} = \bigoplus_{k=0}^{\infty} \mathcal{P}_k$$

with $\mathcal{P}_k$ the eigenspace of $\mathbb{E}$ corresponding to the eigenvalue k. It is moreover easily seen that $\mathcal{P}_k$ is a finite dimensional vectorspace with dimension

$$\dim \mathcal{P}_k = \sum_{i=0}^{min(k,2n)} \binom{2n}{i} \binom{k-i+m-1}{m-1}. \tag{2.5}$$

We can also define the fermionic and bosonic Gamma operators

$$\begin{aligned}\Gamma_b &= -\underline{x}\partial_{\underline{x}} - \mathbb{E}_b\\ \Gamma_f &= \underline{x\grave{}}\partial_{\underline{x\grave{}}} - \mathbb{E}_f\end{aligned}$$

and the super Gamma operator as

$$\Gamma = x\partial_x - \mathbb{E}.$$

It is important to note that $\Gamma \neq \Gamma_b + \Gamma_f$. More precisely we have that

$$
\begin{aligned}
\Gamma &= x\partial_x - \mathbb{E}\\
&= (\underline{x} + \underline{\grave{x}})(\partial_{\underline{\grave{x}}} - \partial_{\underline{x}}) - \mathbb{E}_b - \mathbb{E}_f\\
&= (\underline{\grave{x}}\partial_{\underline{\grave{x}}} - \mathbb{E}_f) + (-\underline{x}\partial_{\underline{x}} - \mathbb{E}_b) + (\underline{x}\partial_{\underline{\grave{x}}} - \underline{\grave{x}}\partial_{\underline{x}})\\
&= \Gamma_f + \Gamma_b + B_x.
\end{aligned}
$$

The operator $B_x = \underline{x}\partial_{\underline{\grave{x}}} - \underline{\grave{x}}\partial_{\underline{x}}$ will be called the bending operator, since it transforms bosonic vector variables into fermionic vector variables and vice versa:

$$
\begin{aligned}
B(\underline{x}) &= m\underline{\grave{x}}\\
B(\underline{\grave{x}}) &= -2n\underline{x}.
\end{aligned}
$$

Finally we define the (super) Laplace-Beltrami operator as:

$$\Delta_{LB} = (M - 2 - \Gamma)\Gamma$$

by analogy with classical Clifford analysis (see e.g. [38]).

Remark 2. *The framework established in this and the previous section can also be extended to include differential forms and associated contraction operators. As we will not need them in the following chapters, we refer the interested reader to [29, 28].*

2.3 Realization of harmonic and Clifford analysis in superspace

In this section we will first establish the most important commutation relations that exist between the operators introduced in the previous section. Then we will show that we have indeed constructed a representation of harmonic resp. Clifford analysis in superspace.

We begin with the following important technical lemma.

Lemma 1. *The following operator equality holds*

$$x\partial_x + \partial_x x = 2\mathbb{E} + M.$$

Proof. We calculate the operator $\partial_x x$ as

$$\begin{aligned}
\partial_x x &= (\partial_{\underline{\grave{x}}} - \partial_{\underline{x}})(\underline{x} + \underline{\grave{x}})\\
&= \partial_{\underline{\grave{x}}}\underline{x} + \partial_{\underline{\grave{x}}}\underline{\grave{x}} - \partial_{\underline{x}}\underline{x} - \partial_{\underline{x}}\underline{\grave{x}}\\
&= -\underline{x}\partial_{\underline{\grave{x}}} + \partial_{\underline{\grave{x}}}\underline{\grave{x}} - \partial_{\underline{x}}\underline{x} + \underline{\grave{x}}\partial_{\underline{x}}.
\end{aligned}$$

We then immediately obtain

$$\begin{aligned}
\underline{x}\partial_{\underline{\grave{x}}} &= -\partial_{\underline{\grave{x}}}\underline{x}\\
\underline{\grave{x}}\partial_{\underline{x}} &= -\partial_{\underline{x}}\underline{\grave{x}}.
\end{aligned}$$

We also have

$$\begin{aligned}
\partial_{\underline{x}}\underline{x} &= \sum_{i,j=1}^{m} e_j \partial_{x_j} e_i x_i\\
&= \sum_{i,j=1}^{m} e_j e_i (\delta_{ij} + x_i \partial_{x_j})\\
&= -m - \mathbb{E}_b - \sum_{i,j=1; i\neq j}^{m} e_i e_j x_i \partial_{x_j}\\
&= -m - 2\mathbb{E}_b - \underline{x}\partial_{\underline{x}}
\end{aligned}$$

and similarly

$$\begin{aligned}
\partial_{\underline{\grave{x}}}\underline{\grave{x}} &= 2\sum_{j=1}^{n}(\grave{e}_{2j}\partial_{\grave{x}_{2j-1}} - \grave{e}_{2j-1}\partial_{\grave{x}_{2j}})\sum_{i=1}^{n}(\grave{e}_{2i-1}\grave{x}_{2i-1} + \grave{e}_{2i}\grave{x}_{2i})\\
&= 2\sum_{i\neq j}(\grave{e}_{2j}\partial_{\grave{x}_{2j-1}} - \grave{e}_{2j-1}\partial_{\grave{x}_{2j}})(\grave{e}_{2i-1}\grave{x}_{2i-1} + \grave{e}_{2i}\grave{x}_{2i})\\
&\quad +2\sum_{i}\left(\grave{e}_{2i}\partial_{\grave{x}_{2i-1}} - \grave{e}_{2i-1}\partial_{\grave{x}_{2i}}\right)\left(\grave{e}_{2i-1}\grave{x}_{2i-1} + \grave{e}_{2i}\grave{x}_{2i}\right)\\
&= -2\sum_{i\neq j}(\grave{e}_{2i-1}\grave{x}_{2i-1} + \grave{e}_{2i}\grave{x}_{2i})(\grave{e}_{2j}\partial_{\grave{x}_{2j-1}} - \grave{e}_{2j-1}\partial_{\grave{x}_{2j}})\\
&\quad +2\sum_{i}\left(\grave{e}_{2i}\partial_{\grave{x}_{2i-1}} - \grave{e}_{2i-1}\partial_{\grave{x}_{2i}}\right)\left(\grave{e}_{2i-1}\grave{x}_{2i-1} + \grave{e}_{2i}\grave{x}_{2i}\right)\\
&= -2\sum_{i,j}(\grave{e}_{2i-1}\grave{x}_{2i-1} + \grave{e}_{2i}\grave{x}_{2i})(\grave{e}_{2j}\partial_{\grave{x}_{2j-1}} - \grave{e}_{2j-1}\partial_{\grave{x}_{2j}}) - 2n + 2\mathbb{E}_f\\
&= -\underline{\grave{x}}\partial_{\underline{\grave{x}}} - 2n + 2\mathbb{E}_f
\end{aligned}$$

from which the lemma follows. □

Now we can formulate the most important intertwining relations between the operators ∂_x, x, $\mathbb{E}$ and Γ.

Theorem 1 (Commutation rules)**.** *The operators x, ∂_x, $\mathbb{E}$ and Γ show the following properties*

1.
$$\begin{cases} x\partial_x + \partial_x x &= 2\mathbb{E} + M \\ x\partial_x - \partial_x x &= 2\Gamma - M \end{cases}$$

2.
$$\begin{cases} [\mathbb{E}, x] &= x \\ [\mathbb{E}, \partial_x] &= -\partial_x \end{cases}$$

3.
$$\begin{cases} [\mathbb{E}, \Gamma] &= 0 \\ [x\partial_x, \partial_x x] &= 0 \end{cases}$$

4.
$$\begin{cases} \Gamma x^{2k} &= x^{2k}\Gamma \\ \Gamma x^{2k+1} &= x^{2k+1}(M - 1 - \Gamma) \end{cases}$$

5.
$$\begin{cases} \Gamma \partial_x^{2k} &= \partial_x^{2k}\Gamma \\ \Gamma \partial_x^{2k+1} &= \partial_x^{2k+1}(M - 1 - \Gamma). \end{cases}$$

Proof.

1. The first relation follows from lemma 1. The second one is calculated as follows:

$$\begin{aligned} x\partial_x - \partial_x x &= 2x\partial_x - (x\partial_x + \partial_x x) \\ &= 2x\partial_x - 2\mathbb{E} - M \\ &= 2\Gamma - M. \end{aligned}$$

2. This can be found by direct calculation. Alternatively, it is sufficient to let the operators act on an arbitrary $F \in \mathcal{P} \otimes \mathcal{C}$ with $\mathbb{E}F = kF$. We then find

$$\begin{aligned} [\mathbb{E}, x]\, F &= \mathbb{E}xF - x\mathbb{E}F \\ &= (k+1)xF - kxF \\ &= xF. \end{aligned}$$

The second relation is obtained in a similar manner.

3. $[\mathbb{E}, \Gamma]$ is calculated in the same way as in 2. We also find

$$\begin{array}{rcl}[x\partial_x, \partial_x x] & = & [\mathbb{E}+\Gamma, \mathbb{E}-\Gamma+M] \\ & = & 0.\end{array}$$

4. We calculate Γx

$$\begin{array}{rcl}\Gamma x & = & (x\partial_x - \mathbb{E})x \\ & = & x\partial_x x - \mathbb{E}x \\ & = & x(2\mathbb{E}+M-x\partial_x) - \mathbb{E}x \\ & = & x(\mathbb{E}+M-\Gamma) - \mathbb{E}x \\ & = & [x, \mathbb{E}] + x(M-\Gamma) \\ & = & x(M-1-\Gamma).\end{array}$$

Iteration of this formula yields the desired result.

5. Similar to the previous calculation.

□

As a consequence we obtain

Corollary 1. *One has*

$$\begin{array}{rcl}\Gamma\Delta & = & \Delta\Gamma \\ \Gamma\Delta_{LB} & = & \Delta_{LB}\Gamma.\end{array}$$

Using theorem 1, we can now prove that the Laplace-Beltrami operator really is a scalar operator. This can be seen from the following calculation:

$$\begin{array}{rcl}\Delta_{LB} & = & \Gamma(M-2-\Gamma) \\ & = & (x\partial_x - \mathbb{E})(M-2-x\partial_x+\mathbb{E}) \\ & = & x\partial_x(M-2-x\partial_x+\mathbb{E}) + \mathbb{E}x\partial_x - \mathbb{E}(M-2+\mathbb{E}) \\ & = & (M-2)x\partial_x + x\partial_x\mathbb{E} + \mathbb{E}x\partial_x - x(2\mathbb{E}+M-x\partial_x)\partial_x - \mathbb{E}(M-2+\mathbb{E}) \\ & = & x^2\partial_x^2 - \mathbb{E}(M-2+\mathbb{E}) - 2x\partial_x + x\mathbb{E}\partial_x - x\partial_x + x\partial_x + x\mathbb{E}\partial_x - x2\mathbb{E}\partial_x \\ & = & x^2\partial_x^2 - \mathbb{E}(M-2+\mathbb{E}),\end{array}$$

yielding

$$\Delta_{LB} = x^2\Delta - \mathbb{E}(M-2+\mathbb{E})$$

which is indeed a scalar operator. Moreover it may be obtained from the classical expression of the Laplace-Beltrami operator by substituting M for m.

We shall now obtain the first important theorem of this chapter, proving that we have indeed obtained a representation of harmonic analysis in superspace.

Theorem 2. *The operators $\Delta/2$, $x^2/2$ and $\mathbb{E}+M/2$ generate the Lie algebra $\mathfrak{sl}_2(\mathbb{R})$.*

Proof. We have

$$\begin{array}{rcl} \left[\Delta/2, x^2/2\right] & = & \mathbb{E}+M/2 \\ \left[\Delta/2, \mathbb{E}+M/2\right] & = & 2\Delta/2 \\ \left[x^2/2, \mathbb{E}+M/2\right] & = & -2x^2/2 \end{array}$$

which are the canonical commutation relations of $\mathfrak{sl}_2(\mathbb{R})$, see e.g. [55, 65]. $\square$

Remark 3. *The $\mathfrak{sl}_2(\mathbb{R})$-relations given in theorem 2 are of course also valid for the purely bosonic case ($n=0$), as we are then reduced to the classical Laplace operator. More surprisingly, the same relations remain valid if one deforms the Laplacian to the so-called Dunkl Laplacian (see e.g. [48, 50]), a second order operator which contains a combination of differential and difference operators in such a way that the $O(m)$-symmetry of the classical Laplacian is broken to a finite subgroup $\mathcal{G}$ of $O(m)$. We refer the reader to [62] and [6] for more details.*

Now let us consider the algebra generated by x and ∂_x. We then obtain the following theorem.

Theorem 3. *The operators ∂_x and x generate a finite-dimensional Lie superalgebra, isomorphic to $\mathfrak{osp}(1|2)$.*

Proof. We clearly have that

$$\begin{array}{rcl} \{x, x\} & = & 2x^2 \\ \{\partial_x, \partial_x\} & = & 2\Delta \\ \{\partial_x, x\} & = & 2\mathbb{E}+M \end{array}$$

using lemma 1. In theorem 2 we have already proven that x^2, Δ and $\mathbb{E}+M/2$ generate $\mathfrak{sl}_2(\mathbb{R})$. The mixed commutators are given by

$$\begin{array}{lcl@{\qquad}lcl} \left[x, x^2\right] & = & 0 & \left[\partial_x, x^2\right] & = & 2x \\ \left[x, \Delta\right] & = & -2\partial_x & \left[\partial_x, \Delta\right] & = & 0 \\ \left[x, \mathbb{E}+M/2\right] & = & -x & \left[\partial_x, \mathbb{E}+M/2\right] & = & \partial_x \end{array}$$

proving that we have a finite-dimensional Lie superalgebra. If we normalize the operators to

$$\frac{1}{2}\Delta, -\frac{1}{2}x^2, \frac{1}{2}(\mathbb{E} + M/2), \frac{1}{2\sqrt{2}}ix, -\frac{1}{2\sqrt{2}}i\partial_x$$

we obtain the standard commutation relations for $\mathfrak{osp}(1|2)$ (see e.g. [54]). $\square$

Remark 4. *There also exists a refinement of the theory of the Dunkl Laplacian to Clifford analysis (see [18]). It can be proven that the basic operators in this framework also satisfy the* $\mathfrak{osp}(1|2)$ *commutation relations (see [76]).*

Chapter 3

Spherical harmonics and monogenics

In this chapter we will study spherical harmonics and monogenics, defined as polynomial null-solutions of the super Laplace, resp. Dirac operator. After some definitions and technical results, we will first study the so-called Fischer decomposition, which gives a decomposition of the space of polynomials in the commuting and anti-commuting variables in terms of spaces of spherical harmonics multiplied with even powers of the vector variable x. We will also determine projection operators selecting the different components in this decomposition. Next we will study spherical harmonics in more detail. We will show that the natural group-action to be considered is that of $SO(m) \times Sp(2n)$. As spaces of spherical harmonics in superspace are not irreducible under this action, we will present the complete decomposition. Finally we will refine the results obtained to spherical monogenics and construct a basis using a Cauchy-Kowalewskaia extension technique.

The results of this chapter can be found in the papers [31, 32, 26].

3.1 Definitions and preliminaries

We define spherical harmonics in superspace as homogeneous polynomial null-solutions of the super Laplace operator.

Definition 1. *A (super) spherical harmonic of degree k is an element $H_k \in \mathcal{P}$*

satisfying

$$\begin{array}{rcl}\Delta H_k &=& 0\\ \mathbb{E}H_k &=& kH_k, \quad i.e.\ H_k \in \mathcal{P}_k.\end{array}$$

The space of spherical harmonics of degree k will be denoted by $\mathcal{H}_k$.

In the purely bosonic case (i.e. the case with only commuting variables) we denote $\mathcal{H}_k$ by $\mathcal{H}_k^b$, in the purely fermionic case (the case with only anti-commuting variables) by $\mathcal{H}_k^f$. If necessary for clarity, the variables under consideration will be mentioned also, e.g. $\mathcal{H}_k^f(x\grave{}_1, \ldots, x\grave{}_{2n})$.

In a similar way we define spherical monogenics as homogeneous polynomial null-solutions of the super Dirac operator.

Definition 2. *A (super) spherical monogenic of degree k is an element $M_k \in \mathcal{P} \otimes \mathcal{C}$ satisfying*

$$\begin{array}{rcl}\partial_x M_k &=& 0\\ \mathbb{E}M_k &=& kM_k, \quad i.e.\ M_k \in \mathcal{P}_k \otimes \mathcal{C}.\end{array}$$

The space of spherical monogenics of degree k will be denoted by $\mathcal{M}_k$.

In the purely bosonic, resp. fermionic case we will use the notations $\mathcal{M}_k^b$, $\mathcal{M}_k^f$. As $\partial_x^2 = \Delta$ we immediately have that $\mathcal{M}_k \subset \mathcal{H}_k \otimes \mathcal{C}$.

For the sequel we need formulae for the iterated actions of Δ, resp. ∂_x on spaces of the type $x^{2i}\mathcal{H}_k$ resp. $x^j\mathcal{M}_l$. We start with the following lemma.

Lemma 2. *One has the following relations:*

$$\begin{array}{llcl}(i) & \Delta(x^{2t}R_k) &=& 2t(2k+M+2t-2)x^{2t-2}R_k + x^{2t}\Delta R_k\\ (ii) & \Delta^{t+1}(x^2R_{2t}) &=& 4(t+1)(M/2+t)\Delta^t(R_{2t}),\end{array}$$

with $R_i \in \mathcal{P}_i$.

Proof. Using $\left[\Delta, x^2\right] = 4\mathbb{E} + 2M$ we immediately have that

$$\Delta(x^2R_k) = (4k+2M)R_k + x^2\Delta R_k. \tag{3.1}$$

Formula (i) then follows using induction on t.

Iterating formula (3.1) gives

$$\begin{aligned}
\Delta^{t+1}(x^2 R_{2t}) &= \Delta^t \left(2(4t+M)R_{2t} + x^2 \Delta R_{2t}\right) \\
&= 2\left((4t+M) + (4t+M-4) + \ldots + (4+M) + M\right) \\
&\quad \times \Delta^t R_{2t} \\
&= 2(\sum_{i=0}^{t} M + 4\sum_{i=0}^{t}(t-i))\Delta^t R_{2t} \\
&= 2((t+1)M + 4t(t+1) - 4t(t+1)/2)\Delta^t R_{2t} \\
&= 4(t+1)(M/2+t)\Delta^t R_{2t}
\end{aligned}$$

thus proving the second statement. □

Iteration of lemma 2 leads to the following result.

Lemma 3. *Let $H_k \in \mathcal{H}_k$ and $M \notin -2\mathbb{N}$. Then for all $i, j, k \in \mathbb{N}$ one has*

$$\Delta^i(x^{2j} H_k) = \begin{cases} c_{i,j,k} x^{2j-2i} H_k, & i \leq j \\ 0, & i > j \end{cases}$$

with

$$c_{i,j,k} = 4^i \frac{j!}{(j-i)!} \frac{\Gamma(k+M/2+j)}{\Gamma(k+M/2+j-i)}.$$

Proof. Using lemma 2 (i), we have for $H_k \in \mathcal{H}_k$

$$\Delta(x^{2t} H_k) = 2t(2k+M+2t-2)x^{2t-2} H_k.$$

Iteration then yields

$$\begin{aligned}
\Delta^i(x^{2j} H_k) &= 2j(2k+M+2j-2)\Delta^{i-1} x^{2j-2} H_k \\
&= 2j(2j-2)(2k+M+2j-2)(2k+M+2j-4)\Delta^{i-2} x^{2j-4} H_k \\
&= \ldots \\
&= 4^i \frac{j!}{(j-i)!} \frac{\Gamma(k+M/2+j)}{\Gamma(k+M/2+j-i)} x^{2j-2i} H_k
\end{aligned}$$

thus completing the proof. □

If $m = 0$, lemma 3 still holds if $j \leq n-k$. The coefficients $c_{i,j,k}$ then simplify to

$$c_{i,j,k} = 4^i \frac{j!}{(j-i)!} \frac{(n+i-k-j)!}{(n-k-j)!}.$$

In the case of $\mathcal{C}$-valued polynomials we have the following lemma.

Lemma 4. *Let $s \in \mathbb{N}$, then for $R_k \in \mathcal{P}_k \otimes \mathcal{C}$ the following holds*

$$\begin{aligned} \partial_x(x^{2s} R_k) &= 2s x^{2s-1} R_k + x^{2s} \partial_x R_k \\ \partial_x(x^{2s+1} R_k) &= (2k+M+2s) x^{2s} R_k - x^{2s+1} \partial_x R_k. \end{aligned}$$

Proof. Using $\{x, \partial_x\} = 2\mathbb{E} + M$ we find

$$\partial_x(x R_k) = (2k+M) R_k - x \partial_x R_k.$$

The result then follows immediately by an induction argument. □

3.2 The Fischer decomposition

3.2.1 The basic theorem

In this section we give the decomposition of the space $\mathcal{P}_k$ in terms of the spaces $x^{2i}\mathcal{H}_{k-2i}$, a decomposition which is well-known in the purely bosonic case (see e.g. [2, 102]). In our case we have the following result.

Theorem 4 (Fischer decomposition)**.** *If $M \notin -2\mathbb{N}$, $\mathcal{P}_k$ decomposes as*

$$\mathcal{P}_k = \bigoplus_{i=0}^{\lfloor \frac{k}{2} \rfloor} x^{2i} \mathcal{H}_{k-2i}. \tag{3.2}$$

If $m = 0$ then one has

$$\begin{aligned} \mathcal{P}_k &= \bigoplus_{i=0}^{\lfloor \frac{k}{2} \rfloor} \underline{x}^{2i} \mathcal{H}^f_{k-2i}, \quad k \leq n & (3.3) \\ \mathcal{P}_{2n-k} &= \bigoplus_{i=0}^{\lfloor \frac{k}{2} \rfloor} \underline{x}^{2n-2k+2i} \mathcal{H}^f_{k-2i}, \quad k \leq n. & (3.4) \end{aligned}$$

Proof. There are several different proofs possible of the statement in the theorem. We present here an induction argument for the case $M \notin -2\mathbb{N}$. First note that the theorem holds trivially for $\mathcal{P}_0$ and $\mathcal{P}_1$. We then use induction on the degree. To simplify notations we only consider the case $\mathcal{P}_{2k}$, the case $\mathcal{P}_{2k+1}$ being completely similar. So consider $p \in \mathcal{P}_{2k}$. We then have that $\Delta p \in \mathcal{P}_{2k-2}$. Applying the induction hypothesis we have

$$\Delta p = \sum_{i=0}^{k-1} x^{2i} H_{2k-2-2i}, \quad H_{2k-2-2i} \in \mathcal{H}_{2k-2-2i}.$$

Each term in this summation can be integrated with respect to the Laplace operator. Indeed, due to lemma 2 we have

$$\Delta x^{2i+2}\mathcal{H}_k = (2i+2)(2k+M+2i)x^{2i}\mathcal{H}_k.$$

This allows us to write p as

$$p = \sum_{i=0}^{k-1} \frac{x^{2i+2} H_{2k-2-2i}}{(2i+2)(4k-4-2i+M)} + H_{2k}$$

with H_{2k} some element of $\mathcal{H}_{2k}$. The restriction $M \notin -2\mathbb{N}$ is necessary to guarantee that the denominator is different from zero.

Now we only need to prove the uniqueness of this decomposition. Suppose there exist $H_i \in \mathcal{H}_i$ such that $\sum_{i=0}^{k} x^{2i} H_{2k-2i} = 0$. Acting on this equation with Δ^k yields $H_0 = 0$. Similarly, acting with Δ^{k-1} yields $H_2 = 0$. Continuing in this way we find $H_i = 0$ and the uniqueness is proven.

Now we consider the purely fermionic case $m = 0$. The proof of the decomposition for $\mathcal{P}_k$, $k \leq n$ is the same as above, because in this case the denominator is also different from zero. Now use the fact that as vectorspaces $\mathcal{P}_k \cong \mathcal{P}_{2n-k}$, an isomorphism being given by multiplication with $\underline{x}\grave{}^{2n-2k}$. Indeed, on each piece of the decomposition of $\mathcal{P}_k$ we have the following equality

$$\Delta_f^{n-k} \underline{x}\grave{}^{2n-2k} \underline{x}\grave{}^{2i} \mathcal{H}_{k-2i}^f = c_{n-k,n-k+i,k-2i} \underline{x}\grave{}^{2i} \mathcal{H}_{k-2i}^f$$

with $c_{n-k,n-k+i,k-2i} \neq 0$, so the map $\underline{x}\grave{}^{2n-2k} : \mathcal{P}_k \to \mathcal{P}_{2n-k}$ can be inverted. □

Remark 5. *Note that if $M \in -2\mathbb{N}$ and $m \neq 0$, there is no Fischer decomposition. This is due to the fact that the spaces $x^{2i}\mathcal{H}_{k-2i}$ are no longer disjoint. If we consider e.g. the case $M = 0$ then $\Delta x^2 = 0$ and we have $x^2 \in \mathcal{H}_2$.*

Remark 6. *There also exists a Fischer decomposition in the case of the Dunkl Laplacian, see [6]. The proof is similar, because it essentially comes down to the action of the $\mathfrak{sl}_2(\mathbb{R})$ algebra generated by the basic operators.*

3.2.2 Graphical representation

First note that, using the Fischer decomposition, the algebra $\mathcal{P}$ decomposes as

$$\begin{aligned}\mathcal{P} &= \bigoplus_{k=0}^{\infty} \mathcal{P}_k \\ &= \bigoplus_{j=0}^{\infty}\bigoplus_{k=0}^{\infty} x^{2j}\mathcal{H}_k.\end{aligned}$$

It is instructive to represent this decomposition in the following diagram

$$\begin{tikzcd}
\mathcal{P}_0 & \mathcal{P}_1 & \mathcal{P}_2 & \mathcal{P}_3 & \mathcal{P}_4 & \mathcal{P}_5 & \mathcal{P}_6 & \cdots \\
\mathcal{H}_0 \arrow[rr] & & x^2\mathcal{H}_0 \arrow[rr] & & x^4\mathcal{H}_0 \arrow[rr] & & x^6\mathcal{H}_0 & \cdots \\
& \mathcal{H}_1 \arrow[rr] & & x^2\mathcal{H}_1 \arrow[rr] & & x^4\mathcal{H}_1 & \cdots & \\
& & \mathcal{H}_2 \arrow[rr] & & x^2\mathcal{H}_2 \arrow[rr] & & x^4\mathcal{H}_2 & \cdots \\
& & & \mathcal{H}_3 \arrow[rr] & & x^2\mathcal{H}_3 & \cdots & \\
& & & & \mathcal{H}_4 \arrow[rr] & & x^2\mathcal{H}_4 & \cdots \\
& & & & & \mathcal{H}_5 & \cdots & \\
& & & & & & \mathcal{H}_6 & \cdots
\end{tikzcd}$$

The k-th column in this diagram gives the direct sum decomposition of $\mathcal{P}_k$. Each row corresponds to an infinite-dimensional representation of $\mathfrak{sl}_2(\mathbb{R})$. The action of the three generators x^2, Δ, $\mathbb{E} + \frac{M}{2}$ of this Lie algebra is given in the following diagram:

$$\begin{tikzcd}
\mathcal{H}_k \arrow[r, shift left, "x^2"] \arrow[loop below, "\mathbb{E}+\frac{M}{2}"] & x^2\mathcal{H}_k \arrow[l, shift left, "\Delta"] \arrow[r, shift left, "x^2"] \arrow[loop below, "\mathbb{E}+\frac{M}{2}"] & x^4\mathcal{H}_k \arrow[l, shift left, "\Delta"] \arrow[r, shift left, "x^2"] \arrow[loop below, "\mathbb{E}+\frac{M}{2}"] & x^6\mathcal{H}_k \arrow[l, shift left, "\Delta"] \arrow[r, shift left, "x^2"] \arrow[loop below, "\mathbb{E}+\frac{M}{2}"] & x^8\mathcal{H}_k \arrow[l, shift left, "\Delta"] \arrow[r, shift left, "x^2"] \arrow[loop below, "\mathbb{E}+\frac{M}{2}"] & \cdots \arrow[l, shift left, "\Delta"]
\end{tikzcd}$$

In the purely fermionic case we have that

$$\mathcal{P} = \bigoplus_{k=0}^{n} \bigoplus_{j=0}^{n-k} \underline{x}^{2j} \mathcal{H}_k^f.$$

Again there is a graphical interpretation, shown in the next diagram (where we have chosen $m = 0$, $n = 4$):

$$\begin{tikzcd}[column sep=small]
\mathcal{P}_0 & \mathcal{P}_1 & \mathcal{P}_2 & \mathcal{P}_3 & \mathcal{P}_4 & \mathcal{P}_5 & \mathcal{P}_6 & \mathcal{P}_7 & \mathcal{P}_8 \\
\mathcal{H}_0^f \arrow[rr] & & \underline{x}^2\mathcal{H}_0^f \arrow[rr] & & \underline{x}^4\mathcal{H}_0^f \arrow[rr] & & \underline{x}^6\mathcal{H}_0^f \arrow[rr] & & \underline{x}^8\mathcal{H}_0^f \\
& \mathcal{H}_1^f \arrow[rr] & & \underline{x}^2\mathcal{H}_1^f \arrow[rr] & & \underline{x}^4\mathcal{H}_1^f \arrow[rr] & & \underline{x}^6\mathcal{H}_1^f & \\
& & \mathcal{H}_2^f \arrow[rr] & & \underline{x}^2\mathcal{H}_2^f \arrow[rr] & & \underline{x}^4\mathcal{H}_2^f & & \\
& & & \mathcal{H}_3^f \arrow[rr] & & \underline{x}^2\mathcal{H}_3^f & & & \\
& & & & \mathcal{H}_4^f. & & & &
\end{tikzcd}$$

The k-th column in this diagram again gives the direct sum decomposition of $\mathcal{P}_k$. Each row now corresponds to a finite-dimensional representation of $\mathfrak{sl}_2(\mathbb{R})$. The action of the Lie algebra is given in the following diagram:

$$\begin{tikzcd}
\mathcal{H}_k^f \arrow[r, shift left, "\underline{x}^2"] \arrow[loop below, "\mathbb{E}-n"] & \underline{x}^2\mathcal{H}_k^f \arrow[l, shift left, "\Delta_f"] \arrow[r, shift left, "\underline{x}^2"] \arrow[loop below, "\mathbb{E}-n"] & \underline{x}^4\mathcal{H}_k^f \arrow[l, shift left, "\Delta_f"] \arrow[r, shift left, "\underline{x}^2"] \arrow[loop below, "\mathbb{E}-n"] & \underline{x}^6\mathcal{H}_k^f \arrow[l, shift left, "\Delta_f"] \arrow[r, shift left, "\underline{x}^2"] \arrow[loop below, "\mathbb{E}-n"] & \cdots \arrow[l, shift left, "\Delta_f"] \arrow[r, shift left, "\underline{x}^2"] & \underline{x}^{2n-2k}\mathcal{H}_k^f. \arrow[l, shift left, "\Delta_f"] \arrow[loop below, "\mathbb{E}-n"]
\end{tikzcd}$$

3.2.3 Projection operators

We can explicitly determine the Fischer decomposition of a given polynomial $R_k \in \mathcal{P}_k$. To that end we have to construct a set of operators $\mathbb{P}_i^k$, $i = 0, \ldots, \lfloor \frac{k}{2} \rfloor$, satisfying:

$$\mathbb{P}_i^k(x^{2j}\mathcal{H}_{k-2j}) = \delta_{ij}\mathcal{H}_{k-2j}.$$

In view of lemma 3 we propose the following form for $\mathbb{P}_i^k$:

$$\mathbb{P}_i^k = \sum_{j=0}^{\lfloor k/2 \rfloor - i} a_j^{i,k} x^{2j} \Delta^{i+j}.$$

It immediately follows that $\mathbb{P}_i^k(x^{2j}\mathcal{H}_{k-2j}) = 0$ for all $j < i$. Now requesting that

$$\mathbb{P}_i^k(x^{2j}\mathcal{H}_{k-2j}) = \delta_{ij}\mathcal{H}_{k-2j}, \quad j \geq i$$

leads to a set of equations allowing to determine the coefficients $a_j^{i,k}$. Expressing that $\mathbb{P}_i^k(x^{2i}\mathcal{H}_{k-2i}) = \mathcal{H}_{k-2i}$ yields

$$a_0^{i,k} = \frac{1}{c_{i,i,k-2i}}.$$

Expressing that $\mathbb{P}_i^k(x^{2i+2}\mathcal{H}_{k-2i-2}) = 0$ similarly yields

$$\begin{aligned} a_1^{i,k} &= -\frac{c_{i,i+1,k-2i-2}}{c_{i+1,i+1,k-2i-2}} a_0^{i,k} \\ &= -\frac{1}{4(k-2i-2+\frac{M}{2})} a_0^{i,k}. \end{aligned}$$

Proceeding in the same way we arrive at the following hypothesis

$$a_l^{i,k} = \frac{(-1)^l}{4^l l!} \frac{\Gamma(k-2i-l-1+M/2)}{\Gamma(k-2i-1+M/2)} a_0^{i,k}$$

which can be proven using induction. Indeed, suppose that the statement holds for $a_j^{i,k}$, $j \leq l$, then we prove that it also holds for $a_{l+1}^{i,k}$. This last coefficient has to satisfy

$$\sum_{j=0}^{l+1} a_j^{i,k}\, c_{i+j,i+l+1,k-2i-2l-2} = 0.$$

Substituting the known expressions we obtain

$$\begin{aligned} a_{l+1}^{i,k} &= -\sum_{j=0}^{l} a_j^{i,k} \frac{c_{i+j,i+l+1,k-2i-2l-2}}{c_{i+l+1,i+l+1,k-2i-2l-2}} \\ &= -\frac{a_0^{i,k}}{4^{l+1}(l+1)!} \frac{\Gamma(k-2i-2l-2+\frac{M}{2})}{\Gamma(k-2i-1+\frac{M}{2})} \\ &\quad \times \sum_{j=0}^{l} (-1)^j \binom{l+1}{j} \frac{\Gamma(k-2i-j-1+\frac{M}{2})}{\Gamma(k-2i-2l-2+\frac{M}{2}+l+1-j)}. \end{aligned}$$

Putting $\alpha = k - 2i + \frac{M}{2} - 1$, we still need to prove that

$$\sum_{j=0}^{l} (-1)^j \binom{l+1}{j} \frac{\Gamma(\alpha-j)}{\Gamma(\alpha-l-j)} = (-1)^l \frac{\Gamma(\alpha-l-1)}{\Gamma(\alpha-2l-1)}$$

or

$$\sum_{j=0}^{l+1}(-1)^j\binom{l+1}{j}\frac{\Gamma(\alpha-j)}{\Gamma(\alpha-l-j)}=0 \tag{3.5}$$

for all α. This is equivalent with proving that the polynomial

$$R(x)=\sum_{j=0}^{l+1}(-1)^j\binom{l+1}{j}(x-j-1)(x-j-2)\ldots(x-j-l)$$

is identically zero for all $l \in \mathbb{N}$, which will be shown in lemma 5. This completes the proof by induction.

Finally we obtain the following expression for the coefficients $a_l^{i,k}$:

$$a_l^{i,k}=\frac{(-1)^l}{4^{l+i}l!i!}(k-2i+M/2-1)\frac{\Gamma(k-2i-l-1+M/2)}{\Gamma(k-i+M/2)}. \tag{3.6}$$

We summarize the result in the following theorem.

Theorem 5. *The operator* $\mathbb{P}_i^k$ *defined by*

$$\mathbb{P}_i^k=\sum_{j=0}^{\lfloor k/2\rfloor-i}a_j^{i,k}x^{2j}\Delta^{i+j}$$

with $a_j^{i,k}$ *as in formula (3.6) satisfies*

$$\mathbb{P}_i^k(x^{2j}\mathcal{H}_{k-2j})=\delta_{ij}\mathcal{H}_{k-2j}.$$

It is also immediately clear that

$$\sum_{i=0}^{\lfloor \frac{k}{2}\rfloor}x^{2i}\mathbb{P}_i^k=\mathrm{id}_{\,\mathcal{P}_k}.$$

Note that the theorem is not valid if $M \in -2\mathbb{N}$ and $m \neq 0$. In that case there is no Fischer decomposition and we end up in the poles of the Gamma function appearing in formula (3.6).

We now give the technical lemma needed to complete the proof of the previous theorem.

Lemma 5. *The polynomial $R(x)$ defined by*

$$R(x) = \sum_{j=0}^{l+1} (-1)^j \binom{l+1}{j} (x-j-1)(x-j-2)\ldots(x-j-l)$$

is identically zero for all positive integers l.

Proof. This is clearly equivalent with proving that

$$\sum_{j=0}^{l+1} (-1)^j \binom{l+1}{j} j^m = 0 \tag{3.7}$$

for all $m \in \mathbb{N}$.

One easily checks that the statement is correct for all l if $m = 0$. Now we proceed by induction. Suppose (3.7) holds for all l and all $m \leq p$, then we prove that it also holds for $m = p+1$. Indeed:

$$\begin{aligned}
\sum_{j=0}^{l+1} (-1)^j \binom{l+1}{j} j^{p+1} &= \sum_{j=1}^{l+1} (-1)^j \binom{l}{j-1} j^p (l+1) \\
&= -(l+1) \sum_{j=1}^{l+1} (-1)^{j-1} \binom{l}{j-1} j^p \\
&= -(l+1) \sum_{j=0}^{l} (-1)^j \binom{l}{j} (j+1)^p.
\end{aligned}$$

This last expression vanishes using the induction hypothesis and the binomial theorem. This completes the proof. $\square$

Remark 7. *Formula (3.5) also follows from the Gauss summation theorem for the hypergeometric series ${}_2F_1$ at the value* 1.

It is also possible to construct the projection operators on the different summands in the Fischer decomposition using a different method. This method is easier but gives a less concrete form of the operator. First we note that the Laplace-Beltrami operator Δ_{LB} commutes with x^2. We then immediately find that

$$\begin{aligned}
\Delta_{LB} x^{2j} \mathcal{H}_k &= x^{2j} \Delta_{LB} \mathcal{H}_k \\
&= -k(M-2+k) x^{2j} \mathcal{H}_k,
\end{aligned}$$

so each summand in the Fischer decomposition is an eigenspace of the Laplace-Beltrami operator. It is easy to see that if $M > 0$, then for each $k \geq 0$ the eigenvalue $-k(M-2+k)$ is different. If $M < 0$ and M is odd, then some values of k give rise to the same eigenvalue. However, it is easily seen that this can only happen if they differ by an odd integer. In the purely fermionic case $(m = 0)$, then again $-k(M-2+k)$ is different for each k because $0 \leq k \leq n$. This means that in the Fischer decomposition of $\mathcal{P}_k$

$$\mathcal{P}_k = \bigoplus_{j=0}^{\lfloor \frac{k}{2} \rfloor} x^{2j} \mathcal{H}_{k-2j}$$

each summand has a different eigenvalue with respect to the Laplace-Beltrami operator. Hence the operator $\widetilde{\mathbb{P}}_i^k$ defined by

$$\widetilde{\mathbb{P}}_i^k = \prod_{l=0,\, l\neq i}^{\lfloor \frac{k}{2} \rfloor} \frac{\Delta_{LB} + (k-2l)(M-2+k-2l)}{2(i-l)(2k-2i-2l+M-2)}$$

satisfies

$$\widetilde{\mathbb{P}}_i^k (x^{2j} \mathcal{H}_{k-2j}) = \delta_{ij} x^{2j} \mathcal{H}_{k-2j}.$$

We also have that $\widetilde{\mathbb{P}}_i^k = x^{2i} \mathbb{P}_i^k$ when acting on $\mathcal{P}_k$.

3.3 Spherical harmonics in superspace

3.3.1 Basic properties

We first prove the following result concerning the surjectivity of the Laplace operator on $\mathcal{P}$.

Theorem 6. *Δ is a surjective operator on $\mathcal{P}$ if $m \neq 0$. If $m = 0$, then Δ is only surjective on $\oplus_{i=0}^n \mathcal{P}_i$.*

Proof. If $M \notin -2\mathbb{N}$ the theorem follows immediately from the Fischer decomposition, as we have that $\Delta(x^{2i+2}\mathcal{H}_k) = (2i+2)(2k+M+2i)x^{2i}\mathcal{H}_k$, allowing us to integrate arbitary polynomials. The same technique works if $m = 0$, but then the surjectivity holds only on $\oplus_{i=0}^n \mathcal{P}_i$ and not on $\oplus_{i=n+1}^{2n} \mathcal{P}_i$.

We have to use a different method for the general case, inspired by [59], lemma 3.1.2. First note that the Laplace operator is clearly surjective on $\mathcal{P}_0$

and $\mathcal{P}_1$. So consider a monomial of the form $x_1^{\alpha_1} \ldots x_m^{\alpha_m} \grave{x}_1^{\beta_1} \ldots \grave{x}_{2n}^{\beta_{2n}}$ with $\alpha_i \in \mathbb{N}$, $\beta_i \in \{0,1\}$ and $\sum \alpha_i + \sum \beta_i = k-2$, $k \geq 2$. It then suffices to show that there exists a polynomial $g \in \mathcal{P}_k$ with $\Delta g = x_1^{\alpha_1} \ldots x_m^{\alpha_m} \grave{x}_1^{\beta_1} \ldots \grave{x}_{2n}^{\beta_{2n}}$. If $\alpha_1 = k-2$ or $\alpha_1 = k-3$ this is certainly the case. We now proceed by induction with respect to descending values of α_1. So we assume that the statement holds for all monomials $x_1^{\alpha_1} \ldots x_m^{\alpha_m} \grave{x}_1^{\beta_1} \ldots \grave{x}_{2n}^{\beta_{2n}}$ with $\gamma_1 < \alpha_1 \leq k-2$ and we show that it also holds for $x_1^{\gamma_1} \ldots x_m^{\alpha_m} \grave{x}_1^{\beta_1} \ldots \grave{x}_{2n}^{\beta_{2n}}$. It is clear that

$$x_1^{\gamma_1} \ldots x_m^{\alpha_m} \grave{x}_1^{\beta_1} \ldots \grave{x}_{2n}^{\beta_{2n}} = -\frac{1}{(\gamma_1+1)(\gamma_1+2)} \Delta \left(x_1^{\gamma_1+2} \ldots x_m^{\alpha_m} \grave{x}_1^{\beta_1} \ldots \grave{x}_{2n}^{\beta_{2n}} \right) + \Sigma(x)$$

with $\Sigma(x)$ a sum of monomials where x_1 has exponent $\gamma_1 + 2$. Hence, there exists a polynomial $r(x)$ such that $\Sigma(x) = \Delta r(x)$ and the theorem follows. $\square$

Using this theorem we can now calculate the dimensions of spaces of spherical harmonics in superspace.

Lemma 6. *One has that*

$$\dim \mathcal{H}_k = \dim \mathcal{P}_k - \dim \mathcal{P}_{k-2}$$

where

$$\dim \mathcal{P}_k = \sum_{i=0}^{min(k,2n)} \binom{2n}{i} \binom{k-i+m-1}{m-1}$$

and by definition $\dim \mathcal{P}_{-1} = \dim \mathcal{P}_{-2} = 0$.

Proof. Immediate, using theorem 6 and formula (2.5). $\square$

In the purely bosonic and the purely fermionic case the previous lemma reduces respectively to

$$\dim \mathcal{H}_k^b = \binom{k+m-1}{m-1} - \binom{k-2+m-1}{m-1}$$

and

$$\begin{aligned} \dim \mathcal{H}_k^f &= \binom{2n}{k} - \binom{2n}{k-2}, \quad k \leq n \\ \dim \mathcal{H}_k^f &= 0, \quad k > n. \end{aligned}$$

3.3.2 Group action

In this section we show that the group $G = SO(m) \times Sp(2n)$ gives the appropriate action on $\mathcal{P}$ for our purposes.

First of all, let us state the properties we want this group to exhibit:

- $G \cdot \mathcal{P}_k \subseteq \mathcal{P}_k$, i.e. the degree of homogeneity is preserved under G
- $G \cdot x^2 = x^2$, i.e. the quadratic polynomial x^2 is invariant under G.

The first property restricts possible transformations to

$$\begin{array}{rcl} y_i & = & \sum_{k=0}^{m} a_k^i x_k + \sum_{l=0}^{2n} b_l^i \grave{x}_l \\ \grave{y}_j & = & \sum_{k=0}^{m} c_k^j x_k + \sum_{l=0}^{2n} d_l^j \grave{x}_l \end{array}$$

with $a_k^i, b_l^i, c_k^j, d_l^j \in \mathbb{R}$. In matrix notation we have that

$$y = Sx = \left(\begin{array}{c|c} A & B \\ \hline C & D \end{array} \right) x$$

with $x = (x_1 \ \ldots \ x_m \mid \grave{x}_1 \ \ldots \ \grave{x}_{2n})^T$.

Similarly

$$x^2 = x^T Q x = x^T \left(\begin{array}{c|c} -1 & 0 \\ \hline 0 & J \end{array} \right) x$$

with

$$J = \frac{1}{2} \begin{pmatrix} 0 & 1 & & & \\ -1 & 0 & & & \\ & & \ddots & & \\ & & & 0 & 1 \\ & & & -1 & 0 \end{pmatrix}.$$

As we want that $y^2 = x^2$, this means that

$$S^T Q S = Q.$$

In terms of A, B, C and D this yields

$$\begin{array}{rcl} -A^T A + C^T J C & = & -1 \qquad (3.8) \\ -A^T B + C^T J D & = & 0 \qquad (3.9) \\ -B^T A + D^T J C & = & 0 \qquad (3.10) \\ -B^T B + D^T J D & = & J. \qquad (3.11) \end{array}$$

Adding to equation (3.8) its transpose and taking into account that $J^T = -J$, leads to $A^T A = 1$. Adding the transpose of (3.9) to (3.10) leads to $B^T A = 0$ and thus to $B = 0$ as A is invertible. As $D^T J D = J$ we have that $D \in Sp(2n)$ and D is invertible. Then (3.10) becomes $D^T J C = 0$ and thus $C = 0$ as also J is invertible. We conclude that the group G can be taken to be $SO(m) \times Sp(2n)$. The Lie algebra for G is then given by the semi-simple algebra $\mathfrak{so}(m) \oplus \mathfrak{sp}_{\mathbb{C}}(2n)$, the irreducible finite-dimensional representations of which are defined as tensor products $\mathbb{V}_\lambda \otimes \mathbb{W}_\mu$, where $\mathbb{V}_\lambda$ denotes the irreducible $\mathfrak{so}(m)$-module with highest weight $\underline{\lambda}$ and $\mathbb{W}_\mu$ the irreducible $\mathfrak{sp}_{\mathbb{C}}(2n)$-module with highest weight $\underline{\mu}$.

It is also easily seen that $G \cdot \mathcal{H}_k \subseteq \mathcal{H}_k$ since the super Laplace operator is invariant under the action of G. However, the spaces $\mathcal{H}_k$ are not irreducible under the action of G. As opposed to the purely bosonic and fermionic case, spaces of homogeneous (polynomial) solutions for the super Laplace operator do not lead to irreducible modules for the Lie algebra underlying the symmetry of the system. The complete decomposition will be presented in theorem 8.

3.3.3 Representations of $SO(m)$ and $Sp(2n)$

Spherical harmonics in $\mathbb{R}^m$

The classical theory of spherical harmonics in $\mathbb{R}^m$ is very well known (see e.g. [2, 102]). In this case the Fischer decomposition takes the following form:

$$\mathcal{P}_k = \bigoplus_{i=0}^{\lfloor \frac{k}{2} \rfloor} \underline{x}^{2i} \mathcal{H}^b_{k-2i},$$

where each space $\mathcal{H}^b_k$ provides a model for the irreducible $\mathfrak{so}(m)$-module with highest weight $(k, 0, \ldots, 0)$.

Moreover, if $m > 2$ this is also the decomposition of the space of homogeneous polynomials of degree k into irreducible pieces under the action of $SO(m)$. In the case where $m = 2$, the spaces $\mathcal{H}^b_{k-2i}$ all have dimension two and are also irreducible, when working over $\mathbb{R}$ as we do here. When working over $\mathbb{C}$, the spaces are still reducible because $SO(2)$ is abelian and all complex irreducible representations of abelian groups are one-dimensional.

Fermionic or symplectic harmonics

In the purely fermionic case the Fischer decomposition is given by

$$
\begin{aligned}
\mathcal{P}_k &= \bigoplus_{i=0}^{\lfloor \frac{k}{2} \rfloor} \underline{\grave{x}}^{2i} \mathcal{H}^f_{k-2i}, \quad k \leq n & (3.12)\\
\mathcal{P}_{2n-k} &= \bigoplus_{i=0}^{\lfloor \frac{k}{2} \rfloor} \underline{\grave{x}}^{2n-2k+2i} \mathcal{H}^f_{k-2i}, \quad k \leq n. & (3.13)
\end{aligned}
$$

Each space $\mathcal{H}^f_k$ provides a model for the fundamental representation for $\mathfrak{sp}_{\mathbb{C}}(2n)$ with highest weight $(1, \ldots, 1, 0, \ldots, 0)$, where the integer 1 is to be repeated k times ($k \leq n$) (see [55]). This means that we have a decomposition of $\mathcal{P}_k$ into irreducible pieces under the action of $Sp(2n)$.

From the fermionic Fischer decomposition we also obtain the following formulae:

$$
\begin{aligned}
\underline{\grave{x}}^{2i} \mathcal{H}^f_k &= 0 \quad \text{for all } i > n-k\\
\underline{\grave{x}}^{2i} \mathcal{H}^f_k &\neq 0 \quad \text{for all } i \leq n-k.
\end{aligned}
$$

Construction of a basis:

It is possible to construct a basis of $\mathcal{H}^f_k$ by decomposing this space under the action of the subgroup $Sp(2) \times Sp(2n-2)$ of $Sp(2n)$. This leads to the following theorem.

Theorem 7. *If $1 < k \leq n$, then the space $\mathcal{H}^f_k(\grave{x}_1, \ldots, \grave{x}_{2n})$ decomposes as*

$$
\begin{aligned}
&\mathcal{H}^f_k(\grave{x}_3, \ldots, \grave{x}_{2n}) \;\oplus\; \mathcal{H}^f_1(\grave{x}_1, \grave{x}_2) \otimes \mathcal{H}^f_{k-1}(\grave{x}_3, \ldots, \grave{x}_{2n})\\
&\oplus \left[\grave{x}_1 \grave{x}_2 + \frac{1}{k-n-1} (\grave{x}_3 \grave{x}_4 + \ldots + \grave{x}_{2n-1} \grave{x}_{2n}) \right] \mathcal{H}^f_{k-2}(\grave{x}_3, \ldots, \grave{x}_{2n}).
\end{aligned}
$$

If $k = 1$, $\mathcal{H}^f_1(\grave{x}_1, \ldots, \grave{x}_{2n})$ decomposes as

$$
\mathcal{H}^f_1(\grave{x}_1, \ldots, \grave{x}_{2n}) = \mathcal{H}^f_1(\grave{x}_3, \ldots, \grave{x}_{2n}) \oplus \mathcal{H}^f_1(\grave{x}_1, \grave{x}_2).
$$

Proof. One easily checks, using lemma 2, that each term at the right-hand side is contained in $\mathcal{H}^f_k(\grave{x}_1, \ldots, \grave{x}_{2n})$. Moreover these three pieces are all disjoint.

The proof is completed if we check that both sides have the same dimension (as vectorspaces). Indeed, the dimension of the right-hand side is:

$$\begin{aligned}\dim RH &= \dim \mathcal{H}_k^f(\grave{x}_3,\ldots,\grave{x}_{2n}) + \dim \mathcal{H}_1^f(\grave{x}_1,\grave{x}_2)\dim \mathcal{H}_{k-1}^f(\grave{x}_3,\ldots,\grave{x}_{2n})\\ &\quad + \dim \mathcal{H}_{k-2}^f(\grave{x}_3,\ldots,\grave{x}_{2n})\\ &= \binom{2n-2}{k} - \binom{2n-2}{k-2} + 2\left(\binom{2n-2}{k-1} - \binom{2n-2}{k-3}\right)\\ &\quad + \binom{2n-2}{k-2} - \binom{2n-2}{k-4}\\ &= \binom{2n}{k} - \binom{2n}{k-2}\end{aligned}$$

after several applications of Pascal's rule. This equals the dimension of the left-hand side. The proof of the second statement is trivial. □

This theorem can be used to construct bases for $\mathcal{H}_k^f(\grave{x}_1,\ldots,\grave{x}_{2n})$ in an iterative way, since $\mathcal{H}_0^f \cong \mathcal{P}_0$ and $\mathcal{H}_1^f \cong \mathcal{P}_1$.

3.3.4 Decomposition in irreducible pieces

In this section the space $\mathcal{H}_k$ will be decomposed into irreducible pieces under the action of the group $G = SO(m) \times Sp(2n)$. In view of the fact that irreducible representations for G, realized within the space $\mathcal{P}_k$ of homogeneous polynomials in both the commuting and anti-commuting variables, are tensor products of spaces of spherical and symplectic harmonics, it is natural to look for a subspace of $\mathcal{H}_k$ inside the direct sum of subspaces of the form $\underline{x}^{2i}\mathcal{H}_p^b \otimes \underline{\grave{x}}^{2j}\mathcal{H}_q^f$, where $(2i+p)+(2j+q)=k$. This is the subject of the following lemma.

Lemma 7. *If $q < n$ and $k < n-q+1$, there exists a unique homogeneous polynomial $f_{k,p,q} = f_{k,p,q}(\underline{x}^2,\underline{\grave{x}}^2)$ of total degree k such that $f_{k,p,q}\mathcal{H}_p^b \otimes \mathcal{H}_q^f \neq 0$ and*

$$\Delta(f_{k,p,q}\mathcal{H}_p^b \otimes \mathcal{H}_q^f) = 0,$$

where the coefficient of $\underline{x}^{2k}$ in $f_{k,p,q}$ is given by

$$\frac{(n-q)!}{\Gamma(\frac{m}{2}+p+k)}.$$

Remark 8. *The restriction $q < n$ is necessary because in the case that $q = n$ all integer powers $\underline{\grave{x}}^{2j}$ will act trivially on the space $\mathcal{H}_n^f$, for $j > 0$. In the same vein, for $q < n$ there can only be a non-trivial action of $\underline{\grave{x}}^{2j}$ on the space $\mathcal{H}_q^f$ as long as $j \leq k$ with $k + q \leq n$. This explains the restricion $k < n - q + 1$.*

Proof. We first treat the case $p = q = 0$. So we look for a polynomial $f_{k,0,0}(\underline{x}^2, \underline{\grave{x}}^2)$ of the following form:

$$f_{k,0,0} = \sum_{i=0}^{k} a_i \underline{x}^{2k-2i} \underline{\grave{x}}^{2i}.$$

We now demand that

$$\Delta(f_{k,0,0}\mathcal{H}_0^b \otimes \mathcal{H}_0^f) = \Delta(f_{k,0,0}) = 0.$$

As $\Delta = \Delta_b + \Delta_f$ we find, using lemma 2, that

$$\begin{aligned}
& \Delta(f_{k,0,0}) \\
= & \sum_{i=0}^{k-1} a_i(2k-2i)(m+2k-2i-2)\underline{x}^{2k-2i-2}\underline{\grave{x}}^{2i} \\
& + \sum_{i=1}^{k} a_i 2i(2i-2-2n)\underline{x}^{2k-2i}\underline{\grave{x}}^{2i-2} \\
= & \; 4\sum_{i=0}^{k-1} \left[a_i(k-i)(\frac{m}{2}+k-i-1) + a_{i+1}(i+1)(i-n)\right] \underline{x}^{2k-2i-2}\underline{\grave{x}}^{2i}.
\end{aligned}$$

Hence we obtain the following recursion relation for the a_i:

$$a_{i+1} = \frac{(k-i)(\frac{m}{2}+k-i-1)}{(n-i)(i+1)} a_i,$$

which leads to the explicit formula

$$a_i = \frac{\Gamma(\frac{m}{2}+k)}{n!} \binom{k}{i} \frac{(n-i)!}{\Gamma(\frac{m}{2}+k-i)} a_0.$$

If we put $a_0 = \frac{n!}{\Gamma(\frac{m}{2}+k)}$ we finally obtain

$$f_{k,0,0} = \sum_{i=0}^{k} \binom{k}{i} \frac{(n-i)!}{\Gamma(\frac{m}{2}+k-i)} \underline{x}^{2k-2i}\underline{\grave{x}}^{2i}. \tag{3.14}$$

The general case is now easily obtained. Indeed, the polynomial $f_{k,p,q}$ satisfying

$$\Delta(f_{k,p,q}\mathcal{H}_p^b \otimes \mathcal{H}_q^f) = 0$$

is found by the following substitutions in formula (3.14): $m \rightarrow m+2p$, $n \rightarrow n-q$ (using lemma 2). This yields

$$f_{k,p,q} = \sum_{i=0}^{k} \binom{k}{i} \frac{(n-q-i)!}{\Gamma(\frac{m}{2}+p+k-i)} \underline{x}^{2k-2i} \underline{\grave{x}}^{2i}.$$

Note that this result again explains the restrictions put on k and q. □

We list some special cases, viz. the polynomials $f_{i,0,0}$ for $i = 1, 2, 3$:

$$\begin{aligned}
f_{1,0,0} &= \frac{n!}{\Gamma(\frac{m}{2}+1)} \left(\underline{x}^2 + \frac{m}{2n} \underline{\grave{x}}^2 \right) \\
f_{2,0,0} &= \frac{n!}{\Gamma(\frac{m}{2}+2)} \left(\underline{x}^4 + \frac{m+2}{n} \underline{x}^2 \underline{\grave{x}}^2 + \frac{m(m+2)}{4n(n-1)} \underline{\grave{x}}^4 \right) \\
f_{3,0,0} &= \frac{n!}{\Gamma(\frac{m}{2}+3)} \left(\underline{x}^6 + \frac{3(m+4)}{2n} \underline{x}^4 \underline{\grave{x}}^2 \right. \\
&\quad \left. + \frac{3(m+2)(m+4)}{4n(n-1)} \underline{x}^2 \underline{\grave{x}}^4 + \frac{m(m+2)(m+4)}{8n(n-1)(n-2)} \underline{\grave{x}}^6 \right).
\end{aligned}$$

Using some elementary identities for special functions, the polynomials obtained in lemma 7 can be rewritten in terms of well-known functions. Indeed, recalling the fact that

$$\Gamma(z)\Gamma(1-z) = \frac{\pi}{\sin \pi z},$$

and referring to the expression of the Jacobi polynomial $P_n^{\alpha,\beta}(t)$ in terms of the hypergeometric function, given by

$$P_n^{\alpha,\beta}(t) = \frac{(\alpha+1)_n}{n!} \, F\left(-n, 1+n+\alpha+\beta; 1+\alpha; \frac{1-t}{2} \right),$$

it is clear that the formal finite sum

$$S(t) = \sum_{i=0}^{k} \binom{k}{i} \frac{\Gamma(1+n-q-i)}{\Gamma\left(p+k+\frac{m}{2}-i\right)} \, t^{2i}$$

can be written as

$$S(t) \quad = \quad -\pi \, \frac{\Gamma(1+n-q)}{\Gamma\left(p+k+\frac{m}{2}\right)} \, F\left(-k, 1-k-p-\frac{m}{2}; q-n; -t^2\right).$$

This means that the polynomial $f_{k,p,q}$ can be represented by

$$f_{k,p,q} = -\pi \, \frac{(n-q)!}{\left(p+k+\frac{m}{2}\right)!} \, \frac{k!}{(q-n)_k} \, \underline{x}^{2k} P_k^{(q-n-1, 1-2k-p-q+n-\frac{m}{2})} \left(1+2\frac{\underline{x}\grave{}^2}{\underline{x}^2}\right).$$

We now check that the following dimension formula holds.

Lemma 8. *One has*

$$\begin{aligned} \dim \mathcal{H}_k \quad &= \quad \sum_{i=0}^{\min(n,k)} \dim \mathcal{H}^b_{k-i} \dim \mathcal{H}^f_i \\ &+ \sum_{j=0}^{\min(n,k-1)-1} \sum_{l=1}^{\min(n-j, \lfloor \frac{k-j}{2} \rfloor)} \dim \mathcal{H}^b_{k-2l-j} \dim \mathcal{H}^f_j . \end{aligned}$$

Proof. The cases $\mathcal{H}_1$ and $\mathcal{H}_2$ are easily verified by explicitly writing the decomposition and plugging in the binomial factors. We then proceed by induction. We restrict ourselves to the case $k \leq n$. We then need to prove that

$$\dim \mathcal{H}_k = \sum_{i=0}^{k} \dim \mathcal{H}^b_{k-i} \dim \mathcal{H}^f_i + \sum_{j=0}^{k-2} \sum_{l=1}^{\lfloor \frac{k-j}{2} \rfloor} \dim \mathcal{H}^b_{k-2l-j} \dim \mathcal{H}^f_j . \tag{3.15}$$

Now suppose the lemma holds for $\mathcal{H}_{k-2}$, i.e.

$$\dim \mathcal{H}_{k-2} = \sum_{i=0}^{k-2} \dim \mathcal{H}^b_{k-i-2} \dim \mathcal{H}^f_i + \sum_{j=0}^{k-4} \sum_{l=1}^{\lfloor \frac{k-j}{2} \rfloor -1} \dim \mathcal{H}^b_{k-2l-j-2} \dim \mathcal{H}^f_j .$$

Using this we can rewrite (3.15) as

$$\dim \mathcal{H}_k = \dim \mathcal{H}_{k-2} + \sum_{i=0}^{k} \dim \mathcal{H}^b_{k-i} \dim \mathcal{H}^f_i .$$

We prove that this formula holds. The left-hand side equals

$$LH = \sum_{i=0}^{k} \binom{2n}{i} \binom{k-i+m-1}{m-1} - \sum_{i=0}^{k-2} \binom{2n}{i} \binom{k-i-2+m-1}{m-1}.$$

The right-hand side equals

$$\begin{aligned}
&\sum_{i=0}^{k-2} \binom{2n}{i} \binom{k-i-2+m-1}{m-1} - \sum_{i=0}^{k-4} \binom{2n}{i} \binom{k-i-4+m-1}{m-1} \\
&+ \sum_{i=0}^{k} \left(\binom{2n}{i} - \binom{2n}{i-2} \right) \left(\binom{k-i+m-1}{m-1} - \binom{k-i-2+m-1}{m-1} \right).
\end{aligned}$$

The second line in this equation is expanded as

$$\begin{aligned}
&\sum_{i=0}^{k} \binom{2n}{i} \binom{k-i+m-1}{m-1} - \sum_{i=0}^{k-2} \binom{2n}{i} \binom{k-i-2+m-1}{m-1} \\
-&\sum_{i=0}^{k-2} \binom{2n}{i} \binom{k-i-2+m-1}{m-1} + \sum_{i=0}^{k-4} \binom{2n}{i} \binom{k-i-4+m-1}{m-1}.
\end{aligned}$$

So by adding all terms we see that the right-hand side equals the left-hand side, thus completing the proof. The case where $k > n$ is treated in a similar fashion. $\square$

Now we are able to obtain the main decomposition of this section. First we introduce the operators

$$\begin{aligned}
\mathbb{Q}_{r,s}^{k} &= \prod_{i=0,\ i \neq k-2r-s}^{k} \frac{\Delta_{LB,b} + i(m-2+i)}{(i-k+2r+s)(k+i-2r-s+m-2)} \\
&\quad \times \prod_{j=0,\ j \neq s}^{\min(n,k)} \frac{\Delta_{LB,f} + j(-2n-2+j)}{(j-s)(j+s-2n-2)},
\end{aligned}$$

with

$$\begin{aligned}
\Delta_{LB,b} &= \underline{x}^2 \Delta_b - \mathbb{E}_b(m-2+\mathbb{E}_b) \\
\Delta_{LB,f} &= \underline{x}^{\grave{2}} \Delta_f - \mathbb{E}_f(-2n-2+\mathbb{E}_f)
\end{aligned}$$

the bosonic resp. fermionic Laplace-Beltrami operator. The decomposition is then given in the following theorem.

Theorem 8 (Decomposition of $\mathcal{H}_k$)**.** *Under the action of $SO(m) \times Sp(2n)$ the space $\mathcal{H}_k$ decomposes as*

$$\mathcal{H}_k = \bigoplus_{i=0}^{\min(n,k)} \mathcal{H}^b_{k-i} \otimes \mathcal{H}^f_i \;\oplus \bigoplus_{j=0}^{\min(n,k-1)-1} \; \bigoplus_{l=1}^{\min(n-j, \lfloor \frac{k-j}{2} \rfloor)} f_{l,k-2l-j,j} \mathcal{H}^b_{k-2l-j} \otimes \mathcal{H}^f_j, \tag{3.16}$$

with $f_{l,k-2l-j,j}$ the polynomials determined in lemma 7.

Moreover, all direct summands in this decomposition are irreducible under the action of $SO(m) \times Sp(2n)$ and one has

$$\mathbb{Q}^k_{r,s} \left(f_{l,k-2l-j,j} \mathcal{H}^b_{k-2l-j} \otimes \mathcal{H}^f_j \right) = \delta_{rl} \delta_{sj} f_{l,k-2l-j,j} \mathcal{H}^b_{k-2l-j} \otimes \mathcal{H}^f_j.$$

Proof. Using lemma 7 we see that the right-hand side is contained in the left-hand side. Moreover we have that all summands are mutually disjoint. Indeed, as the bosonic and fermionic Laplace-Beltrami operators $\Delta_{LB,b}$ and $\Delta_{LB,f}$ both commute with $\underline{x}^2$ and $\underline{x}\grave{}^2$ we have that

$$\begin{aligned}
& \Delta_{LB,b} \Delta_{LB,f} f_{l,k-2l-j,j} \mathcal{H}^b_{k-2l-j} \otimes \mathcal{H}^f_j \\
= \;& f_{l,k-2l-j,j} \Delta_{LB,b} \mathcal{H}^b_{k-2l-j} \otimes \Delta_{LB,f} \mathcal{H}^f_j \\
= \;& (k-2l-j)(m-2+k-2l-j)(j)(-2n-2+j) f_{l,k-2l-j,j} \mathcal{H}^b_{k-2l-j} \otimes \mathcal{H}^f_j
\end{aligned}$$

and hence that

$$\mathbb{Q}^k_{r,s} \left(f_{l,k-2l-j,j} \mathcal{H}^b_{k-2l-j} \otimes \mathcal{H}^f_j \right) = \delta_{rl} \delta_{sj} f_{l,k-2l-j,j} \mathcal{H}^b_{k-2l-j} \otimes \mathcal{H}^f_j$$

proving that all summands are disjoint.

Lemma 8 then shows that the left-hand side and the right-hand side of formula (3.16) have the same dimension, so the decomposition holds. As to the irreducibility, the pieces $f_{l,k-2l-j,j} \mathcal{H}^b_{k-2l-j} \otimes \mathcal{H}^f_j$ clearly transform into themselves under the action of $SO(m) \times Sp(2n)$ and they are irreducible as tensor products of irreducible representations of $SO(m)$ and $Sp(2n)$. ☐

3.4 Spherical monogenics

3.4.1 Basic properties

We start with the following lemma which gives the decomposition of a space of spherical harmonics in terms of spaces of spherical monogenics.

Lemma 9. *If $M \notin -2\mathbb{N}$ or $m = 0$, the space $\mathcal{H}_k \otimes \mathcal{C}$ decomposes as*

$$\mathcal{H}_k \otimes \mathcal{C} = \mathcal{M}_k \oplus x\mathcal{M}_{k-1}.$$

Proof. Let $H_k \in \mathcal{H}_k \otimes \mathcal{C}$. Clearly, it can be decomposed as

$$H_k = \left(H_k - \frac{1}{2k-2+M} x\partial_x H_k \right) + \frac{1}{2k-2+M} x\partial_x H_k$$

and it is easy to check that the first part is an element of $\mathcal{M}_k$ and the second part an element of $x\mathcal{M}_{k-1}$. This decomposition is also unique. Indeed, suppose that $M_k \in \mathcal{M}_k$ and $M_{k-1} \in \mathcal{M}_{k-1}$ satisfy $M_k + xM_{k-1} = 0$. Acting with ∂_x on this equation gives $(2k-2+M)M_{k-1} = 0$ and hence $M_k = M_{k-1} = 0$. □

Now we immediately have the full Fischer decomposition.

Corollary 2 (Fischer decomposition)**.** *If $M \notin -2\mathbb{N}$, $\mathcal{P}_k \otimes \mathcal{C}$ decomposes as*

$$\mathcal{P}_k \otimes \mathcal{C} = \bigoplus_{i=0}^{k} x^i \mathcal{M}_{k-i}. \tag{3.17}$$

If $m = 0$, then the decomposition is given by

$$\mathcal{P}_k \otimes \mathcal{C} = \bigoplus_{i=0}^{k} \underline{\grave{x}}^i \mathcal{M}^f_{k-i}, \quad k \leq n \tag{3.18}$$

$$\mathcal{P}_{2n-k} \otimes \mathcal{C} = \bigoplus_{i=0}^{k} \underline{\grave{x}}^{2n-2k+i} \mathcal{M}^f_{k-i}, \quad k \leq n. \tag{3.19}$$

Remark 9. *Recently, the Fischer decomposition for the Dunkl version of the Dirac operator has been established (see [76]). The authors give a very readable and detailed account of the ideas lying behind this decomposition.*

Remark 10. *There also exists a Fischer decomposition on the level of differential forms (where the Dirac operator is replaced by the Dirac contractor). In the purely bosonic case, this is treated in [38]. The general case is treated in [32].*

It is again possible to represent the decomposition of the algebra $\mathcal{P}\otimes\mathcal{C}$ in a diagram. We obtain

$$\begin{tikzcd}
\mathcal{P}_0\otimes\mathcal{C} & \mathcal{P}_1\otimes\mathcal{C} & \mathcal{P}_2\otimes\mathcal{C} & \mathcal{P}_3\otimes\mathcal{C} \arrow[d, equal] & \mathcal{P}_4\otimes\mathcal{C} & \mathcal{P}_5\otimes\mathcal{C} & \cdots \\
\mathcal{M}_0 \arrow[r] & x\mathcal{M}_0 \arrow[r] & x^2\mathcal{M}_0 \arrow[r] & x^3\mathcal{M}_0 \arrow[r] & x^4\mathcal{M}_0 \arrow[r] & x^5\mathcal{M}_0 & \cdots \\
 & & & \oplus & & & \\
 & \mathcal{M}_1 \arrow[r] & x\mathcal{M}_1 \arrow[r] & x^2\mathcal{M}_1 \arrow[r] & x^3\mathcal{M}_1 \arrow[r] & x^4\mathcal{M}_1 & \cdots \\
 & & & \oplus & & & \\
 & & \mathcal{M}_2 \arrow[r] & x\mathcal{M}_2 \arrow[r] & x^2\mathcal{M}_2 \arrow[r] & x^3\mathcal{M}_2 & \cdots \\
 & & & \oplus & & & \\
 & & & \mathcal{M}_3 \arrow[r] & x\mathcal{M}_3 \arrow[r] & x^2\mathcal{M}_3 & \cdots \\
 & & & & \mathcal{M}_4 \arrow[r] & x\mathcal{M}_4 & \cdots \\
 & & & & & \mathcal{M}_5 & \cdots
\end{tikzcd}$$

where the k-th column gives the direct sum decomposition of $\mathcal{P}_k\otimes\mathcal{C}$. Each row now corresponds to an infinite-dimensional representation of the Lie superalgebra $\mathfrak{osp}(1|2)$. The action of the generators is given by

$$\begin{tikzcd}
\mathcal{M}_k \arrow[r, shift left, "x"] \arrow[loop below, "\mathbb{E}+\frac{M}{2}"] & x\mathcal{M}_k \arrow[l, shift left, "\partial_x"] \arrow[r, shift left, "x"] \arrow[loop below, "\mathbb{E}+\frac{M}{2}"] & x^2\mathcal{M}_k \arrow[l, shift left, "\partial_x"] \arrow[r, shift left, "x"] \arrow[loop below, "\mathbb{E}+\frac{M}{2}"] & x^3\mathcal{M}_k \arrow[l, shift left, "\partial_x"] \arrow[r, shift left, "x"] \arrow[loop below, "\mathbb{E}+\frac{M}{2}"] & x^4\mathcal{M}_k \arrow[l, shift left, "\partial_x"] \arrow[r, shift left, "x"] \arrow[loop below, "\mathbb{E}+\frac{M}{2}"] & \cdots \arrow[l, shift left, "\partial_x"]
\end{tikzcd}$$

In the purely fermionic case the graphical representation is given by (where we have chosen $m=0$, $n=3$):

$$\begin{tikzcd}
\mathcal{P}_0\otimes\mathcal{C} & \mathcal{P}_1\otimes\mathcal{C} & \mathcal{P}_2\otimes\mathcal{C} & \mathcal{P}_3\otimes\mathcal{C} \arrow[d, equal] & \mathcal{P}_4\otimes\mathcal{C} & \mathcal{P}_5\otimes\mathcal{C} & \mathcal{P}_6\otimes\mathcal{C} \\
\mathcal{M}_0^f \arrow[r] & \underline{x}\mathcal{M}_0^f \arrow[r] & \underline{x}^2\mathcal{M}_0^f \arrow[r] & \underline{x}^3\mathcal{M}_0^f \arrow[r] & \underline{x}^4\mathcal{M}_0^f \arrow[r] & \underline{x}^5\mathcal{M}_0^f \arrow[r] & \underline{x}^6\mathcal{M}_0^f \\
 & & & \oplus & & & \\
 & \mathcal{M}_1^f \arrow[r] & \underline{x}\mathcal{M}_1^f \arrow[r] & \underline{x}^2\mathcal{M}_1^f \arrow[r] & \underline{x}^3\mathcal{M}_1^f \arrow[r] & \underline{x}^4\mathcal{M}_1^f & \\
 & & & \oplus & & & \\
 & & \mathcal{M}_2^f \arrow[r] & \underline{x}\mathcal{M}_2^f \arrow[r] & \underline{x}^2\mathcal{M}_2^f & & \\
 & & & \oplus & & & \\
 & & & \mathcal{M}_3^f. & & &
\end{tikzcd}$$

The k-th column again gives the direct sum decomposition of $\mathcal{P}_k \otimes \mathcal{C}$. The k-th row now corresponds to the tensor product of a finite-dimensional representation of the Lie superalgebra $\mathfrak{osp}(1|2)$ with $\mathcal{M}_k^f$. The action of the generators is given by

$$\begin{tikzcd}
\mathcal{M}_k^f \arrow[r, shift left, "\underline{x}"] \arrow[loop below, "\mathbb{E}-n"] & \underline{x}\mathcal{M}_k^f \arrow[l, shift left, "\partial_{\underline{x}}"] \arrow[r, shift left, "\underline{x}"] \arrow[loop below, "\mathbb{E}-n"] & \underline{x}^2\mathcal{M}_k^f \arrow[l, shift left, "\partial_{\underline{x}}"] \arrow[r, shift left, "\underline{x}"] \arrow[loop below, "\mathbb{E}-n"] & \underline{x}^3\mathcal{M}_k^f \arrow[l, shift left, "\partial_{\underline{x}}"] \arrow[r, shift left, "\underline{x}"] \arrow[loop below, "\mathbb{E}-n"] & \cdots \arrow[l, shift left, "\partial_{\underline{x}}"] \arrow[r, shift left, "\underline{x}"] & \underline{x}^{2n-2k}\mathcal{M}_k^f. \arrow[l, shift left, "\partial_{\underline{x}}"] \arrow[loop below, "\mathbb{E}-n"]
\end{tikzcd}$$

We can again construct projection operators on the different summands in the Fischer decomposition, combining theorem 5 with lemma 9. They can also be constructed in a more elegant way. We first need to study the eigenfunctions of the Gamma operator. We have the following lemma.

Lemma 10. *The space $x^l\mathcal{M}_k$ is an eigenspace of Γ corresponding to the eigenvalue*

$$\psi_{l,k} = \begin{cases} -k & \text{if } l \text{ even} \\ (k+M-1) & \text{if } l \text{ odd.} \end{cases}$$

Proof. We examine the two cases. First take $l = 2s$. Then $\left[\Gamma, x^2\right] = 0$ and

$$\begin{aligned} \Gamma(x^{2s}\mathcal{M}_k) &= x^{2s}\Gamma(\mathcal{M}_k) \\ &= -kx^{2s}\mathcal{M}_k. \end{aligned}$$

Now take $l = 2s+1$. Then using $\Gamma x = x(M-1-\Gamma)$ we find

$$\begin{aligned} \Gamma(x^{2s+1}\mathcal{M}_k) &= x^{2s+1}(M-1-\Gamma)(\mathcal{M}_k) \\ &= (k+M-1)x^{2s+1}\mathcal{M}_k. \end{aligned}$$

□

Now consider the Fischer decomposition of $\mathcal{P}_k \otimes \mathcal{C}$ given by

$$\mathcal{P}_k = \oplus_{i=0}^{k} x^i \mathcal{M}_{k-i}.$$

It is then easy to see that each summand in this decomposition corresponds to a different eigenvalue of Γ, independent of the value of M. Hence, we can use the Gamma operator to construct projection operators on each summand of the decomposition. Indeed, the operator $\mathbb{M}_i^k$ (acting on $\mathcal{P}_k \otimes \mathcal{C}$) defined by

$$\begin{aligned}
\mathbb{M}_i^k &= \prod_{r=0, r\neq i/2}^{\lfloor \frac{k}{2} \rfloor} \frac{\Gamma + k - 2r}{i - 2r} \prod_{s=0}^{\lfloor \frac{k-1}{2} \rfloor} \frac{\Gamma - k + 2s + 2 - M}{-2k + i + 2s + 2 - M}, \quad i \text{ even} \\
&= \prod_{r=0}^{\lfloor \frac{k}{2} \rfloor} \frac{\Gamma + k - 2r}{2k - i + M - 1 - 2r} \prod_{s=0, s\neq \frac{i-1}{2}}^{\lfloor \frac{k-1}{2} \rfloor} \frac{\Gamma - k + 2s + 2 - M}{2s + 1 - i}, \quad i \text{ odd}
\end{aligned}$$

clearly satisfies

$$\mathbb{M}_i^k \left(x^j \mathcal{M}_{k-j} \right) = \delta_{ij} x^j \mathcal{M}_{k-j}.$$

Remark 11. *So far we have studied the action of the Dirac operator and the related operators $\mathbb{E}$ and Γ on $\mathcal{P} \otimes \mathcal{C}$. These operators are examples of Clifford-valued differential operators with polynomials coefficients, i.e. elements of the algebra*

$$\mathcal{D} = Alg(e_i; x_i; \partial_{x_i}; \grave{e}_i; \grave{x}_i; \partial_{\grave{x}_i}).$$

Using the Fischer decomposition of $\mathcal{P} \otimes \mathcal{C}$, it is possible to give a decomposition of $\mathcal{D}$ into spaces of operators transforming different summands of the Fischer decomposition into each other. This has been treated in [100] for the purely bosonic case, but the same techniques remain valid in the general case.

3.4.2 Cauchy-Kowalewskaia extension

In the case $m \neq 0$, there exists an elegant technique to construct a basis of $\mathcal{M}_k$, called Cauchy-Kowalewskaia extension (see also [38]). This amounts to constructing an isomorphism

$$CK : \mathcal{P}_k \otimes \mathcal{C} \bmod x_1 \mathcal{P}_{k-1} \otimes \mathcal{C} \longrightarrow \mathcal{M}_k.$$

Note that by $\mathcal{A}_k = \mathcal{P}_k \otimes \mathcal{C} \bmod x_1 \mathcal{P}_{k-1} \otimes \mathcal{C}$ we mean the elements of $\mathcal{P}_k \otimes \mathcal{C}$ that do not contain x_1. We define this map as follows. Let $f \in \mathcal{A}_k$, then

$$CK(f) = \sum_{i=0}^{k} \frac{x_1^i}{i!} (-e_1 \widetilde{\partial_x})^i f$$

with $\widetilde{\partial_x} = \partial_x + e_1 \partial_{x_1}$.

First of all, we have to check that $CK(f) \in \mathcal{M}_k$. To this end it suffices to prove that $\partial_x(CK(f)) = 0$ or

$$\begin{aligned}\partial_x(CK(f)) = 0 &\iff (\widetilde{\partial_x} - e_1\partial_{x_1})(CK(f)) = 0\\ &\iff (e_1\widetilde{\partial_x} + \partial_{x_1})(CK(f)) = 0.\end{aligned}$$

We now calculate

$$\begin{aligned}(e_1\widetilde{\partial_x} + \partial_{x_1})(CK(f)) &= -\sum_{i=0}^{k} \frac{x_1^i}{i!}(-e_1\widetilde{\partial_x})^{i+1} f + \sum_{i=1}^{k} \frac{x_1^{i-1}}{(i-1)!}(-e_1\widetilde{\partial_x})^i f\\ &= -\frac{x_1^k}{k!}(-e_1\widetilde{\partial_x})^{k+1} f\\ &= 0\end{aligned}$$

where the last term vanishes because $f \in \mathcal{P}_k \otimes \mathcal{C}$.

We also define a map $R : \mathcal{M}_k \longrightarrow \mathcal{A}_k$ by

$$R(g) = g \bmod x_1\mathcal{P}_{k-1} \otimes \mathcal{C}.$$

It is easy to see that $R(CK(f)) = f$. We can now state the main theorem of this section

Theorem 9. *CK is an isomorphism (of right $\mathcal{C}$-modules) between $\mathcal{A}_k$ and $\mathcal{M}_k$, with inverse R.*

Proof. It is clear that both CK and R are right $\mathcal{C}$-module morphisms. Because $R(CK(f)) = f$ it follows that CK is injective. We now prove that $CK(R(g)) = g$. It suffices to prove that R is injective. Indeed, from $R(CK(R(g)) - g) = 0$ the statement then follows.

So we prove that R is injective. Suppose $F \in \mathcal{M}_k$ and $R(F) = 0$. Because of the second condition we can expand F as

$$F = x_1h_{k-1} + x_1^2h_{k-2} + \ldots + x_1^kh_0, \quad h_i \in \mathcal{A}_i.$$

Expressing the fact that $\partial_xF = 0$ leads to the following set of equations (where we have collected the terms corresponding to the same power of x_1)

$$\begin{aligned}
-e_1 h_{k-1} &= 0\\
-2e_1 h_{k-2} + \partial_x h_{k-1} &= 0\\
-3e_1 h_{k-3} + \partial_x h_{k-2} &= 0\\
&\vdots\\
-ke_1 h_0 + \partial_x h_1 &= 0.
\end{aligned}$$

The only solution of this system is $h_0 = h_1 = \ldots = h_{k-1} = 0$, which proves the injectivity of R. □

We can now use the previous theorem to construct a basis for $\mathcal{M}_k$ as a right $\mathcal{C}$-module. Indeed, first we construct a basis for $\mathcal{A}_k$. It is easily seen that the set of all monomials

$$x^{\alpha,\beta} = x_2^{\alpha_2} \ldots x_m^{\alpha_m} \grave{x}_1^{\beta_1} \ldots \grave{x}_{2n}^{\beta_{2n}}$$

with $\alpha_i \in \mathbb{N}$, $\beta_i \in \{0,1\}$ and $\sum \alpha_i + \sum \beta_i = k$ is a basis for this space.

So a basis of $\mathcal{M}_k$ is given by the set

$$CK(x^{\alpha,\beta}) = CK(x_2^{\alpha_2} \ldots x_m^{\alpha_m} \grave{x}_1^{\beta_1} \ldots \grave{x}_{2n}^{\beta_{2n}}), \quad \sum \alpha_i + \sum \beta_i = k.$$

Example 1. *As an example, we calculate a basis of $\mathcal{M}_1$. We find the following set of $m + 2n - 1$ polynomials*

$$\begin{aligned}
CK(x_i) &= x_i + e_1 e_i x_1, \quad i = 2, \ldots, m\\
CK(\grave{x}_{2j-1}) &= \grave{x}_{2j-1} - 2e_1 \grave{e}_{2j} x_1, \quad j = 1, \ldots, n\\
CK(\grave{x}_{2j}) &= \grave{x}_{2j} + 2e_1 \grave{e}_{2j-1} x_1, \quad j = 1, \ldots, n.
\end{aligned}$$

Now we can also calculate the dimensions of the spaces $\mathcal{M}_k$. First we note that the dimension (i.e. the rank as a free $\mathcal{C}$-module) of $\mathcal{P}_k \otimes \mathcal{C}$ is given by

$$\dim \mathcal{P}_k \otimes \mathcal{C} = \sum_{i=0}^{min(k,2n)} \binom{2n}{i} \binom{k-i+m-1}{m-1}. \tag{3.20}$$

This follows immediately from the fact that the set

$$x^{\alpha,\beta} = x_1^{\alpha_1} \ldots x_m^{\alpha_m} \grave{x}_1^{\beta_1} \ldots \grave{x}_{2n}^{\beta_{2n}}$$

with $\alpha_i \in \mathbb{N}$, $\beta_i \in \{0,1\}$ and $\sum \alpha_i + \sum \beta_i = k$ is a basis for this space.

Now we use theorem 9 which gives us the dimension of $\mathcal{M}_k$:

$$\dim \mathcal{M}_k = \sum_{i=0}^{min(k,2n)} \binom{2n}{i} \binom{k-i+m-2}{m-2}$$

where we had to substitute m for $m-1$ in formula (3.20).

Chapter 4

Hermite and Gegenbauer polynomials in superspace

An important tool in the study of Clifford analysis are orthogonal polynomials. A generalization of e.g. the Hermite polynomials to Clifford analysis in $\mathbb{R}^m$ was first introduced by Sommen in [94] using the technique of CK or monogenic extension. Recall that the classical Hermite polynomials $H_n(x)$ are given by the generating function

$$e^{2tx-t^2} = \sum_{n=0}^{\infty} H_n(x) \frac{t^n}{n!}$$

which can be rewritten as

$$e^{z^2} = \sum_{n=0}^{\infty} e^{x^2} H_n(ix) \frac{t^n}{n!}, \quad z = x + it. \tag{4.1}$$

In other words, the Hermite polynomials naturally appear when calculating the holomorphic extension of the Gaussian e^{x^2} in a special way. Similarly, in Clifford analysis generalized Hermite polynomials appear when calculating the monogenic extension of the Gaussian $\exp(\underline{x}^2)$ in $\mathbb{R}^m$ to $\mathbb{R}^{m+1}$. In fact, it is even more general to consider the monogenic extension of

$$e^{\underline{x}^2} M_k(x)$$

with $M_k(x) \in \mathcal{M}_k^b$ a spherical monogenic of a certain degree k. This is still in correspondence with formula (4.1), as the one-dimensional Dirac operator only has the constants as polynomial null-solutions. The monogenic extension of $e^{\underline{x}^2} M_k(x)$ to $\mathbb{R}^{m+1}$ is given by

$$\sum_{k=0}^{\infty} \frac{x_{m+1}^k}{k!} (e_{m+1}\partial_{\underline{x}})^k e^{\underline{x}^2} M_k(x)$$

and leads to the following Rodrigues formula, defining the Clifford-Hermite polynomials:

$$H_{k,m}(M_k)(x) = e^{-\underline{x}^2} \partial_{\underline{x}}^k (e^{\underline{x}^2} M_k(x)).$$

Also generalized Gegenbauer polynomials can be introduced in the framework of Clifford analysis, as is done in the Ph.D. thesis of Cnops ([20]). In this thesis, it is also shown that only these two sets of orthogonal polynomials can be extended in a natural way to the framework of Clifford analysis.

For an overview of the properties of the Clifford-Hermite and the Clifford-Gegenbauer polynomials we refer the reader to a.o. [94, 20, 38]. They also have some interesting applications, e.g. in multi-dimensional wavelet analysis (see [14, 15, 12] and [35]).

In the present chapter we want to generalize the Clifford-Hermite and the Clifford-Gegenbauer polynomials to the superspace setting. This is done by a careful analysis of their definition in the Euclidean case, providing us with a canonical way to generalize them. It will turn out that the only difference is that we have to replace the Euclidean dimension by the super-dimension.

Note that also other sets of special polynomials have been introduced in the framework of superspaces. We refer the reader to [41, 44, 43, 42, 45, 46] for polynomials related to solutions of supersymmetric (this is the case $m = 2n$) Calogero-Moser-Sutherland systems. It is interesting to note that the authors are considering symmetric polynomials in superspace, i.e. polynomials invariant under the simultaneous exchange $x_i \leftrightarrow x_j$ and $\grave{x}_i \leftrightarrow \grave{x}_j$.

The results of this chapter have been published in [30].

4.1 Clifford-Hermite polynomials in superspace

In classical Clifford analysis, the Clifford-Hermite polynomials can be defined using the following inner product (if we follow the development in [38])

$$(f, g) = \int_{\mathbb{R}^m} \overline{f(\underline{x})} g(\underline{x}) e^{\underline{x}^2} dV(\underline{x})$$

on $L_2(\mathbb{R}^m; e^{\underline{x}^2})$, where $\bar{.}$ is the main anti-involution on the Clifford algebra $\mathbb{R}_{0,m}$, defined by

$$\begin{aligned}\overline{e_i} &= -e_i\\ \overline{ab} &= \bar{b}\bar{a}, \quad \text{for all } a, b \in \mathbb{R}_{0,m}.\end{aligned}$$

For our purpose, it suffices to know this inner product for functions of the form $f = \underline{x}^s M_k$, $g = \underline{x}^t M_l$ with $M_k \in \mathcal{M}_k^b$ and $M_l \in \mathcal{M}_l^b$ spherical monogenics of degree k, respectively l, in $\mathbb{R}^m$. The previous integral can then be rewritten, using spherical co-ordinates $\underline{x} = r\underline{\xi}$, as

$$\begin{aligned}(\underline{x}^s M_k, \underline{x}^t M_l) &= \int_{\mathbb{R}^m} \overline{M_k}\bar{\underline{x}}^s \underline{x}^t M_l e^{\underline{x}^2} dV(\underline{x})\\ &= \int_0^\infty r^k r^s r^t r^l e^{-r^2} r^{m-1} dr \int_{\mathbb{S}^{m-1}} \overline{M_k(\underline{\xi})}\bar{\underline{\xi}}^s \underline{\xi}^t M_l(\underline{\xi}) d\sigma(\underline{\xi})\\ &= \frac{1}{2}\Gamma(\frac{k+s+t+l+m}{2}) \int_{\mathbb{S}^{m-1}} \overline{M_k(\underline{\xi})}\bar{\underline{\xi}}^s \underline{\xi}^t M_l(\underline{\xi}) d\sigma(\underline{\xi})\end{aligned}$$

with $\Gamma(.)$ denoting the Gamma function. Note that this inner product consists of two parts: a radial part and an angular part which is an integration over the unit-sphere. If we consider e.g. the case $s = 2a$, $t = 2b$ the angular integral simplifies to

$$\int_{\mathbb{S}^{m-1}} \overline{M_k(\underline{\xi})}\bar{\underline{\xi}}^{2a} \underline{\xi}^{2b} M_l(\underline{\xi}) d\sigma(\underline{\xi}) = (-1)^{a+b} \int_{\mathbb{S}^{m-1}} \overline{M_k(\underline{\xi})} M_l(\underline{\xi}) d\sigma(\underline{\xi}).$$

The remaining integral is an inner product on the space of spherical monogenics and can be left out of our discussion, as we will consider a fixed spherical monogenic in the sequel.

So, by introducing the following real vector space in the super setting

$$R(M_k) = \left\{ \sum_{j=0}^{n} a_j x^j M_k \;\mid\; n \in \mathbb{N},\, a_j \in \mathbb{R} \right\}$$

with x the vector variable and where M_k is a spherical monogenic of degree k, fixed once and for all, one can define a bilinear form on $R(M_k)$. This is done by using the previous calculations, however replacing the Euclidean dimension m by the super-dimension M, and leads to the following definition.

Definition 3. *Let $2\beta = M + 2k$, then the bilinear form $\langle \cdot, \cdot \rangle$ on $R(M_k)$ is defined by*

$$\begin{array}{lcl} \langle x^{2s} M_k, x^{2t} M_k \rangle & = & (-1)^{s+t} \frac{1}{2} \Gamma(s+t+\beta) \\ \langle x^{2s+1} M_k, x^{2t} M_k \rangle & = & 0 \\ \langle x^{2s} M_k, x^{2t+1} M_k \rangle & = & 0 \\ \langle x^{2s+1} M_k, x^{2t+1} M_k \rangle & = & (-1)^{s+t} \frac{1}{2} \Gamma(s+t+\beta+1) \end{array}$$

extended by linearity to the whole of $R(M_k)$.

Note that this bilinear form is symmetric, but in general not positive definite (this is only the case if $M \in \mathbb{N}$, $M > 0$). Furthermore it is not defined if and only if $M \in -2\mathbb{N}$, due to the singularities of the Gamma function.

We now introduce the following operator

$$D_+ = \partial_x + 2x$$

which satisfies $D_+(R(M_k)) \subset R(M_k)$ because of lemma 4. Now we have the following important property of $\langle \cdot, \cdot \rangle$.

Proposition 1. *The operators ∂_x and D_+ are dual with respect to $\langle \cdot, \cdot \rangle$, i.e.*

$$\langle D_+ p_i M_k, p_j M_k \rangle = \langle p_i M_k, \partial_x p_j M_k \rangle,$$

with $p_i M_k,\ p_j M_k \in R(M_k)$, where p_i and p_j are polynomials in the vector variable x.

Proof. In [38] the similar proposition in the standard Clifford analysis case is proven by using Stokes' theorem in $\mathbb{R}^m$. In our case we need a different approach.

We have that

$$\langle D_+ x^{2s} M_k, x^{2t} M_k \rangle \ = \ 0 \ = \ \langle x^{2s} M_k, \partial_x x^{2t} M_k \rangle$$

and

$$\begin{array}{cl} & \langle D_+ x^{2s+1} M_k, x^{2t} M_k \rangle \\ = & (2k+2s+M) \langle x^{2s} M_k, x^{2t} M_k \rangle + 2 \langle x^{2s+2} M_k, x^{2t} M_k \rangle \\ = & (2k+2s+M)(-1)^{s+t} \dfrac{1}{2} \Gamma(s+t+\beta) \\ & +2(-1)^{s+t+1} \dfrac{1}{2} \Gamma(s+t+\beta+1) \\ = & (-1)^{s+t} \dfrac{1}{2} \Gamma(s+t+\beta) \left(2k+2s+M-2(s+t+\beta)\right) \end{array}$$

$$\begin{aligned} &= -2t(-1)^{s+t}\frac{1}{2}\Gamma(s+t+\beta) \\ &= \langle x^{2s+1}M_k, 2tx^{2t-1}M_k\rangle \\ &= \langle x^{2s+1}M_k, \partial_x x^{2t}M_k\rangle. \end{aligned}$$

The expression $\langle D_+ x^{2s}M_k, x^{2t+1}M_k\rangle$ is calculated in the same way. □

Now we arrive at the definition of the Clifford-Hermite polynomials in superspace.

Definition 4. *Let M_k be a spherical monogenic of degree k. Then*

$$H_{t,M}(M_k)(x) = (D_+)^t M_k$$

is a Clifford-Hermite polynomial of degree (t,k).

We have that, by lemma 4, $H_{t,M}(M_k)(x) = H_{t,M,k}(x)M_k$, where $H_{t,M,k}(x)$ is a polynomial in the vector variable x, which does not depend on the specific choice of M_k, but only on the integer k. So clearly $H_{t,M}(M_k)(x) \in R(M_k)$.

The first few Clifford-Hermite polynomials have the following general form:

$$\begin{aligned} H_{0,M}(M_k)(x) &= M_k \\ H_{1,M}(M_k)(x) &= 2xM_k \\ H_{2,M}(M_k)(x) &= [4x^2 + 2(2k+M)]M_k \\ H_{3,M}(M_k)(x) &= [8x^3 + 4(2k+M+2)x]M_k \\ H_{4,M}(M_k)(x) &= [16x^4 + 16(2k+M+2)x^2 + 4(2k+M+2)(2k+M)]M_k. \end{aligned}$$

From these examples and the definition of D_+, it follows immediately that if t is even, the polynomials $H_{t,M,k}(x)$ only contain even powers of x and if t is odd, the polynomials $H_{t,M,k}(x)$ only contain odd powers of x.

Now we derive the basic properties of these new polynomials. We first have the following straightforward recursion formula:

Theorem 10 (Recursion formula)**.**

$$H_{t,M}(M_k)(x) = D_+ H_{t-1,M}(M_k)(x).$$

The Clifford-Hermite polynomials are orthogonal with respect to $\langle\cdot,\cdot\rangle$, as is expressed in the following theorem.

Theorem 11 (Orthogonality relation)**.** *If $s \neq t$ then*

$$\langle H_{s,M}(M_k)(x), H_{t,M}(M_k)(x)\rangle = 0.$$

Proof. Suppose $s > t$. Then

$$\begin{aligned}\langle H_{s,M}(M_k)(x), H_{t,M}(M_k)(x)\rangle &= \langle D_+^s M_k, H_{t,M}(M_k)(x)\rangle\\ &= \langle M_k, \partial_x^s H_{t,M}(M_k)(x)\rangle\\ &= 0,\end{aligned}$$

by proposition 1 and lemma 4. □

Lemma 11. *The functions $H_{j,M}(M_k)(x)$, $j = 0, 1, 2, \ldots$ constitute a basis for $R(M_k)$.*

Proof. It suffices to note that the coefficient of $H_{j,M}(M_k)(x)$ in x^j is always different from zero. □

The Clifford-Hermite polynomials are solutions of a partial differential equation in superspace. This equation is given in the following theorem.

Theorem 12 (Differential equation)**.** *$H_{t,M}(M_k)(x)$ is a solution of the following differential equation:*

$$\partial_x^2 H_{t,M}(M_k)(x) + 2x\partial_x H_{t,M}(M_k)(x) - C(t, M, k)H_{t,M}(M_k)(x) = 0$$

with

$$C(t, M, k) = \begin{cases} 2t, & t \text{ even}\\ 2(t + M + 2k - 1), & t \text{ odd.}\end{cases}$$

Proof. This theorem can be proven by induction. This is necessary in case $M \in -2\mathbb{N}$. In the other cases it is also possible to use the method described in [38].

We write the following expansion of the Clifford-Hermite polynomials

$$\begin{aligned}H_{2t,M}(M_k) &= \textstyle\sum_{i=0}^t a_{2i}^{2t} x^{2i} M_k\\ H_{2t+1,M}(M_k) &= \textstyle\sum_{i=0}^t a_{2i+1}^{2t+1} x^{2i+1} M_k.\end{aligned}$$

The recursion formula combined with lemma 4 leads to the following relation among the coefficients

$$\begin{aligned}a_{2i}^{2t} &= (2i + 2k + M)a_{2i+1}^{2t-1} + 2a_{2i-1}^{2t-1}\\ a_{2i+1}^{2t+1} &= (2i + 2)a_{2i+2}^{2t} + 2a_{2i}^{2t}.\end{aligned}$$

We need to prove the following (which is easily seen to be true if $t = 0$)

$$\begin{array}{lcl} \partial_x H_{2t,M}(M_k) & = & 4t H_{2t-1,M}(M_k) \\ \partial_x H_{2t+1,M}(M_k) & = & 2(2t+2k+M) H_{2t,M}(M_k). \end{array} \tag{4.2}$$

or, in terms of the a_j^i,

$$\begin{array}{lcl} 2i a_{2i}^{2t} & = & 4t a_{2i-1}^{2t-1} \\ (2k+2i+M) a_{2i+1}^{2t+1} & = & 2(2t+2k+M) a_{2i}^{2t}. \end{array}$$

Indeed, letting act D_+ on (4.2) then yields the theorem.

Suppose now that formula (4.2) holds for $H_{t,M}(M_k)(x)$, $t \leq 2s$. We show that it also holds for $t = 2s+1$. Indeed,

$$\begin{array}{lcl} (2k+2i+M) a_{2i+1}^{2s+1} & = & (2k+2i+M)((2i+2) a_{2i+2}^{2s} + 2 a_{2i}^{2s}) \\ & = & (2k+2i+M)(4s a_{2i+1}^{2s-1} + 2 a_{2i}^{2s}) \\ & = & 4s a_{2i}^{2s} - 8s a_{2i-1}^{2s-1} + 2(2k+2i+M) a_{2i}^{2s} \\ & = & 2(2s+2k+M) a_{2i}^{2s} + 4i a_{2i}^{2s} - 8s a_{2i-1}^{2s-1} \\ & = & 2(2s+2k+M) a_{2i}^{2s}. \end{array}$$

Similarly we can prove that if the theorem holds for $t \leq 2s+1$, then it also holds for $t = 2s+2$. $\square$

The previous proof can be used to give explicit formulae for the coefficients a_j^i in the expansion of the Hermite polynomials. This yields the following result.

Theorem 13 (Explicit form)**.** *If $M \notin -2\mathbb{N}$ or $m = 0$, then the coefficients in the expansion of the Clifford-Hermite polynomials take the following form*

$$\begin{array}{lcl} a_{2i}^{2t} & = & 2^{2t} \begin{pmatrix} t \\ i \end{pmatrix} \dfrac{\Gamma(t+k+M/2)}{\Gamma(i+k+M/2)} \\ a_{2i+1}^{2t+1} & = & 2^{2t+1} \begin{pmatrix} t \\ i \end{pmatrix} \dfrac{\Gamma(t+1+k+M/2)}{\Gamma(i+1+k+M/2)}. \end{array}$$

Proof. We first prove the formula for a_{2i}^{2t}. We have that, using the expressions from the previous proof

$$\begin{array}{lcl} a_{2i}^{2t} & = & \dfrac{2t}{i} a_{2i-1}^{2t-1} \\ & = & \dfrac{4t(t+k-1+M/2)}{i(i+k-1+M/2)} a_{2i-2}^{2t-2} \end{array}$$

$$\begin{aligned}
&= \ldots \\
&= 2^{2i}\frac{t\ldots(t-i+1)}{i(i-1)\ldots 1}\frac{(t+k-1+M/2)\ldots(t+k-i+M/2)}{(i+k-1+M/2)\ldots(k+M/2)}a_0^{2t-2i} \\
&= 2^{2i}\begin{pmatrix} t \\ i \end{pmatrix}\frac{\Gamma(t+k+M/2)\Gamma(k+M/2)}{\Gamma(t+k-i+M/2)\Gamma(i+k+M/2)}a_0^{2t-2i}.
\end{aligned}$$

So we need a formula for a_0^{2t}. This can be done as follows

$$\begin{aligned}
a_0^{2t} &= (2k+M)a_1^{2t-1} \\
&= (2k+M)2\frac{2t+M+2k-2}{2k+M}a_0^{2t-2} \\
&= 4(t+k-1+M/2)a_0^{2t-2} \\
&= \ldots \\
&= 2^{2t}(t+k-1+M/2)\ldots(k+M/2)a_0^0 \\
&= 2^{2t}\frac{\Gamma(t+k+M/2)}{\Gamma(k+M/2)}.
\end{aligned}$$

Combining these results gives the desired formula for a_{2i}^{2t}. The formula for a_{2i+1}^{2t+1} follows from the observation that

$$a_{2i+1}^{2t+1} = 2\frac{2t+2k+M}{2i+2k+M}a_{2i}^{2t}.$$

□

Now, using the results on the differential equation of the Clifford-Hermite polynomials, we can obtain a second recursion formula, which is a typical three-term recursion relation.

Theorem 14 (Recursion formula bis)**.**

$$H_{t+1,M}(M_k) = 2xH_{t,M}(M_k) + C(t,M,k)H_{t-1,M}(M_k).$$

Proof.

$$\begin{aligned}
H_{t+1,M}(M_k) &= D_+H_{t,M}(M_k) \\
&= (\partial_x + 2x)H_{t,M}(M_k) \\
&= 2xH_{t,M}(M_k) + C(t,M,k)H_{t-1,M}(M_k).
\end{aligned}$$

□

One can also obtain a Rodrigues formula in superspace. First we define the generalized Gaussian function by

$$\exp(x^2) = \sum_{k=0}^{\infty} \frac{1}{k!} x^{2k}.$$

We then have the following theorem.

Theorem 15 (Rodrigues formula)**.** *The Clifford-Hermite polynomials take the form*

$$\begin{aligned} H_{t,M}(M_k)(x) &= \exp(-x^2)(\partial_x)^t \exp(x^2) M_k \\ &= \exp(-x^2/2)(\partial_x + x)^t \exp(x^2/2) M_k. \end{aligned}$$

Proof. This follows immediately from the following operator equalities on $R(P_k)$:

$$\exp(-x^2)\partial_x \exp(x^2) = D_+ = \exp(-x^2/2)(\partial_x + x)\exp(x^2/2)$$

combined with the definition of the Clifford-Hermite polynomials. □

Finally, as $H_{t,M}(M_k)(x) = H_{t,M,k}(x) M_k$ where $H_{t,M,k}(x)$ is a polynomial in the vector variable x, it is a natural question to ask whether these polynomials are related to orthogonal polynomials on the real line. This is indeed the case. More specifically we have the following relations.

Theorem 16. *One has that, if $M \notin -2\mathbb{N}$ or $m = 0$,*

$$\begin{aligned} H_{2t,M,k}(x) &= 2^{2t} t! L_t^{\frac{M}{2}+k-1}(-x^2) \\ H_{2t+1,M,k}(x) &= 2^{2t+1} t! x L_t^{\frac{M}{2}+k}(-x^2), \end{aligned}$$

where L_n^{α} are the generalized Laguerre polynomials on the real line.

Proof. This follows immediately by comparing the coefficients given in theorem 13 with the definition of the generalized Laguerre polynomials:

$$L_t^{\alpha}(x) = \sum_{i=0}^{t} \frac{\Gamma(t+\alpha+1)}{i!(t-i)!\Gamma(i+\alpha+1)} (-x)^i.$$

□

Remark 12. *If we consider the case $m = 1$, $n = 0$ (this is the real line), we can only consider $k = 0$ as there are no polynomial null-solutions of the one-dimensional Dirac operator $i\partial_{x_1}$ except for the constants. In that case the theorem reduces to*

$$\begin{aligned} H_{2t,1,0}(x_1) &= 2^{2t} t! L_t^{-\frac{1}{2}}(x_1^2) \\ H_{2t+1,1,0}(x_1) &= 2^{2t+1} t! i x_1 L_t^{\frac{1}{2}}(x_1^2), \end{aligned}$$

which is, up to multiplication with a minus sign and the complex unit, the relation satisfied by the classical Hermite polynomials on the real line.

Let us finally calculate the normalization constants of the Clifford-Hermite polynomials.

Theorem 17. *One has that*

$$\begin{aligned} \langle H_{2t,M}(M_k)(x), H_{2t,M}(M_k)(x)\rangle &= \frac{1}{2} 4^{2t} t! \Gamma(t + M/2 + k) \\ \langle H_{2t+1,M}(M_k)(x), H_{2t+1,M}(M_k)(x)\rangle &= \frac{1}{2} 4^{2t+1} t! \Gamma(t + M/2 + k + 1). \end{aligned}$$

Proof. We only prove the first relation, the second one being similar. We have that

$$\begin{aligned} & \langle H_{2t,M}(M_k)(x), H_{2t,M}(M_k)(x)\rangle \\ = & \frac{1}{C(2t, M, k)} \langle D_+ \partial_x H_{2t,M}(M_k)(x), H_{2t,M}(M_k)(x)\rangle \\ = & \frac{1}{C(2t, M, k)} \langle \partial_x H_{2t,M}(M_k)(x), \partial_x H_{2t,M}(M_k)(x)\rangle \\ = & C(2t, M, k) \langle H_{2t-1,M}(M_k)(x), H_{2t-1,M}(M_k)(x)\rangle \\ = & \ldots \\ = & C(2t, M, k) C(2t-1, M, k) \ldots C(1, M, k) \langle M_k, M_k\rangle \\ = & C(2t, M, k) C(2t-1, M, k) \ldots C(1, M, k) \frac{1}{2} \Gamma(\beta). \end{aligned}$$

Substituting the actual values for the coefficients $C(i, M, k)$ gives the desired formula. $\square$

4.2 Bases and other versions of the Clifford-Hermite polynomials

In this section we introduce several other sets of polynomials which are related to the Clifford-Hermite polynomials of the previous section and we construct appropriate bases of function spaces, needed in the sequel.

First note that in the case where $M \notin -2\mathbb{N}$ the Clifford-Hermite polynomials form a basis for $\mathcal{P} \otimes \mathcal{C}$ because of the Fischer decomposition. Consequently, the ($\mathcal{C}$-valued) functions $\phi_{j,k,l}(x)$ defined by

$$\begin{aligned} \phi_{j,k,l}(x) &= (\partial_x + x)^j M_k^{(l)} \exp x^2/2 \\ &= H_{j,M,k}(x) M_k^{(l)} \exp x^2/2, \end{aligned}$$

with $M_k^{(l)}$ a basis of $\mathcal{M}_k$, indexed by l, form a basis for the function space $\mathcal{S}(\mathbb{R}^m)_{m|2n}$. In the case $m = 0$, the functions $\phi_{j,k,l}(x)$ also form a basis of $\mathcal{P} \otimes \mathcal{C}$, but now with the indices j and k restricted to the values $j = 0, \ldots, 2n - 2k$ and $k = 0, \ldots, n$. We will refer to them as the Clifford-Hermite functions.

We can also construct a scalar version of the Clifford-Hermite polynomials. If we take the square of the operator D_+ we obtain

$$\begin{aligned} (D_+)^2 &= (\partial_x + 2x)^2 \\ &= \Delta + 4x^2 + 2\{\partial_x, x\} \\ &= \Delta + 4x^2 + 2(2\mathbb{E} + M) \end{aligned}$$

which is a scalar differential operator. It hence makes sense to let it act on a spherical harmonic H_k instead of on a spherical monogenic. This leads us immediately to the following definition.

Definition 5. *Let H_k be a spherical harmonic of degree k. Then*

$$CH_{2t,M}(H_k)(x) = (D_+)^{2t} H_k$$

is a Clifford-Hermite polynomial of degree $(2t, k)$.

The set of polynomials $CH_{2t,M}(H_k)(x)$ clearly satisfies similar properties as the Clifford-Hermite polynomials introduced in the previous section. The precise form of the polynomials depends only on the degree of the spherical harmonic H_k, so we can write $CH_{2t,M}(H_k)(x) = CH_{2t,M,k}(x) H_k$. The Rodrigues formula is given by

$$CH_{2t,M}(H_k)(x) = \exp(-x^2/2)(\partial_x + x)^{2t} \exp(x^2/2) H_k$$

where $(\partial_x + x)^2 = \Delta + x^2 + 2\mathbb{E} + M$. The relation with the generalized Laguerre polynomials is now given by

$$CH_{2t,M,k}(x) = 2^{2t} t! L_t^{\frac{M}{2}+k-1}(-x^2). \tag{4.3}$$

In a similar way as in theorem 12, we may also prove the following theorem.

Theorem 18. *The polynomials $CH_{2t,M}(H_k)(x)$ are eigenfunctions of the operator $\Delta + 2\mathbb{E}$ corresponding to the eigenvalue $2(2t+k)$.*

We also have that if $M \notin -2\mathbb{N}$ the (scalar) functions $\psi_{j,k,l}(x)$ defined by

$$\begin{aligned} \psi_{j,k,l}(x) &= (\partial_x + x)^{2j} H_k^{(l)} \exp x^2/2 \\ &= CH_{2j,M,k}(x) H_k^{(l)} \exp x^2/2, \end{aligned}$$

with $H_k^{(l)}$ a basis of $\mathcal{H}_k$, indexed by l, form a basis for the space $\mathcal{S}(\mathbb{R}^m) \otimes \Lambda_{2n}$. If $m = 0$, the $\psi_{j,k,l}(x)$ form a basis of Λ_{2n}, but with the indices restricted to $j = 0, \ldots, n-k$ and $k = 0, \ldots, n$.

Remark 13. *The interested reader should compare the definition of the polynomials $CH_{2t,M}(H_k)(x)$ with the generalized Hermite polynomials introduced in the context of Dunkl operators (see [86]). It may be shown that the polynomials introduced there satisfy a similar relation as in formula (4.3), but with M replaced by a suitable dimension parameter in the theory of Dunkl operators.*

The previously introduced Clifford-Hermite functions have an important physical application: they are eigenfunctions of a harmonic oscillator in superspace (see section 4.4). For that reason, they are in the case $m = 1$, $n = 0$ sometimes called the physical version of the Hermite polynomials.

In the sequel we will however also need a rescaled version of them (the probabilistic version of the Hermite polynomials), defined by

$$\widetilde{H}_{t,M}(M_k)(x) = \exp(-x^2/2)(\partial_x)^t \exp(x^2/2) M_k$$

with $M_k \in \mathcal{M}_k$, or by

$$\widetilde{CH}_{2t,M}(H_k)(x) = \exp(-x^2/2)(\partial_x)^{2t} \exp(x^2/2) H_k$$

with $H_k \in \mathcal{H}_k$.

It is now easy to check that (in the scalar case) they have the following explicit form

$$\widetilde{CH}_{2t,M,k}(x) = \sum_{i=0}^{t} 2^{2t-2i} \left(\begin{array}{c} t \\ i \end{array} \right) \frac{\Gamma(t+k+M/2)}{\Gamma(i+k+M/2)} x^{2i} \tag{4.4}$$

which means that in terms of Laguerre polynomials we have

$$\widetilde{CH}_{2t,M,k}(x) = 2^{2t} t! L_t^{\frac{M}{2}+k-1}(-x^2/4).$$

Remark 14. *The polynomials $x^t M_k$ and $\widetilde{H}_{t,M}(M_k)$ (with $M_k \in \mathcal{M}_k$) both satisfy the following property:*

$$\partial_x x^t M_k = \begin{cases} t x^{t-1} M_k & t \text{ even} \\ (t-1+2k+M) x^{t-1} M_k & t \text{ odd} \end{cases}$$

$$\partial_x \widetilde{H}_{t,M}(M_k) = \begin{cases} t \widetilde{H}_{t-1,M}(M_k) & t \text{ even} \\ (t-1+2k+M) \widetilde{H}_{t-1,M}(M_k) & t \text{ odd.} \end{cases}$$

This means that both sets of polynomials satisfy the same relation with respect to the action of the Dirac operator. Moreover, if $n = 0$, $m = 1$ and consequently $k = 0$, they both constitute a so-called Appell sequence (see [87]). The present framework hence gives a natural extension of such a sequence to higher dimensions and it would be worthwhile to further analyze this correspondence.

4.3 Clifford-Gegenbauer polynomials in superspace

In the Euclidean case the Clifford-Gegenbauer polynomials are defined making use of the following inner product on the unit ball $B(1)$ in $\mathbb{R}^m$ (see [38])

$$(f,g)_\alpha = \int_{B(1)} \overline{f(\underline{x})} g(\underline{x})(1+\underline{x}^2)^\alpha dV(\underline{x}).$$

However, similar to section 4.1, it suffices to have a computation of this inner product for $f = \underline{x}^s M_k$, $g = \underline{x}^t M_l$ with $M_k \in \mathcal{M}_k^b$ and $M_l \in \mathcal{M}_l^b$ spherical monogenics of degree k respectively l in $\mathbb{R}^m$. In that case the previous integral reduces, using spherical co-ordinates, to

$$\begin{aligned}
(\underline{x}^s M_k, \underline{x}^t M_l)_\alpha &= \int_{B(1)} \overline{M_k}\,\overline{\underline{x}^s}\,\underline{x}^t M_l (1+\underline{x}^2)^\alpha dV(\underline{x}) \\
&= \int_0^1 r^k r^s r^t r^l (1-r^2)^\alpha r^{m-1} dr \int_{\mathbb{S}^{m-1}} \overline{M_k(\underline{\xi})}\,\overline{\underline{\xi}}^s \underline{\xi}^t M_l(\underline{\xi}) d\sigma(\underline{\xi}) \\
&= \frac{1}{2} B(\frac{k+s+t+l+m}{2}, \alpha+1) \int_{\mathbb{S}^{m-1}} \overline{M_k(\underline{\xi})}\,\overline{\underline{\xi}}^s \underline{\xi}^t M_l(\underline{\xi}) d\sigma(\underline{\xi})
\end{aligned}$$

with $B(x,y) = \Gamma(x)\Gamma(y)/\Gamma(x+y)$ the Beta-function.

Again this inner product consists of two parts: a radial part and an angular part which is an inner product on the unit sphere. This second part is treated in the same way as in section 4.1. Restricting ourselves to spaces of the type $R(M_k)$ as in section 4.1 we are thus lead to the following definition, where we have again replaced the Euclidean dimension m by the super-dimension M.

Definition 6. *Let $2\beta = M + 2k$, then the bilinear form $\langle \cdot, \cdot \rangle_\alpha$ (parametrized by α) is defined by*

$$\begin{aligned}
\langle x^{2s} M_k, x^{2t} M_k \rangle_\alpha &= (-1)^{s+t} \tfrac{1}{2} B(s+t+\beta, \alpha+1) \\
\langle x^{2s+1} M_k, x^{2t} M_k \rangle_\alpha &= 0 \\
\langle x^{2s} M_k, x^{2t+1} M_k \rangle_\alpha &= 0 \\
\langle x^{2s+1} M_k, x^{2t+1} M_k \rangle_\alpha &= (-1)^{s+t} \tfrac{1}{2} B(s+t+\beta+1, \alpha+1)
\end{aligned}$$

extended by linearity to the whole of $R(M_k)$.

This bilinear form is well-defined if and only if $\alpha \notin -\mathbb{N}$ and $M \notin -2\mathbb{N}$.

Now we introduce the following important operator

$$D_\alpha = (1+x^2)\partial_x + 2(1+\alpha)x,$$

which satisfies $D_\alpha(R(M_k)) \subset R(M_k)$ because of lemma 4. This operator behaves well with respect to the bilinear form $\langle \cdot, \cdot \rangle_\alpha$ as is shown in the following proposition.

Proposition 2. *The operators ∂_x and D_α are dual with respect to $\langle \cdot, \cdot \rangle_\alpha$, i.e.*

$$\langle D_\alpha p_i M_k, p_j M_k \rangle_\alpha = \langle p_i M_k, \partial_x p_j M_k \rangle_{\alpha+1},$$

with $p_i M_k$, $p_j M_k \in R(M_k)$, where p_i and p_j are polynomials in the vector variable x.

Proof. It suffices to prove the proposition for $\langle D_\alpha x^{2s+1} M_k, x^{2t} M_k\rangle_\alpha$, $\langle D_\alpha x^{2s} M_k, x^{2t+1} M_k\rangle_\alpha$, $\langle D_\alpha x^{2s+1} M_k, x^{2t+1} M_k\rangle_\alpha$ and $\langle D_\alpha x^{2s} M_k, x^{2t} M_k\rangle_\alpha$. We only calculate the first one, the others being completely similar.

$$
\begin{aligned}
& \langle D_\alpha x^{2s+1} M_k, x^{2t} M_k\rangle_\alpha \\
= \;& 2(\alpha+1)\langle x^{2s+2} M_k, x^{2t} M_k\rangle_\alpha + (2k+2s+M)\langle (1+x^2) x^{2s} M_k, x^{2t} M_k\rangle_\alpha \\
= \;& (-1)^{s+t+1}\frac{1}{2}(2\alpha+2+2k+M+2s) B(s+t+\beta+1, \alpha+1) \\
& +(2k+2s+M)(-1)^{s+t}\frac{1}{2} B(s+t+\beta, \alpha+1) \\
= \;& (-1)^{s+t}\frac{1}{2}\Gamma(\alpha+1)\left(-(2\alpha+2+2k+M+2s)\frac{\Gamma(s+t+\beta+1)}{\Gamma(s+t+\beta+\alpha+2)}\right. \\
& \left. +(2k+2s+M)\frac{\Gamma(s+t+\beta)}{\Gamma(s+t+\beta+\alpha+1)}\right) \\
= \;& (-1)^{s+t}\frac{1}{2}\Gamma(\alpha+1)\frac{\Gamma(s+t+\beta)}{\Gamma(s+t+\beta+\alpha+2)}((2k+M+2s)(s+t+\beta+\alpha+1)) \\
& -(2\alpha+2+2k+M+2s)(s+t+\beta) \\
= \;& -(-1)^{s+t}\frac{1}{2}\Gamma(\alpha+1)\frac{\Gamma(s+t+\beta)}{\Gamma(s+t+\beta+\alpha+2)}(\alpha+1)2t \\
= \;& -(-1)^{s+t}\frac{1}{2} B(s+t+\beta, \alpha+2)2t \\
= \;& \langle x^{2s+1} M_k, \partial_x x^{2t} M_k\rangle_{\alpha+1}.
\end{aligned}
$$

□

We are now able to define the Clifford-Gegenbauer polynomials in superspace.

Definition 7. *Let $M_k \in \mathcal{M}_k$ be a spherical monogenic of degree k. Then*

$$C^\alpha_{t,M}(M_k)(x) = D_\alpha D_{\alpha+1} \dots D_{\alpha+t-1} M_k$$

is a Clifford-Gegenbauer polynomial of degree (t,k).

Again we have that, by lemma 4, $C^\alpha_{t,M}(M_k)(x) = C^\alpha_{t,M,k}(x) M_k$, where $C^\alpha_{t,M,k}(x)$ is a polynomial in the vector variable x, which does not depend on M_k, but only on the integer k.

Explicitly, we find the following form for the first Clifford-Gegenbauer polynomials:

$$\begin{array}{rcl} C^{\alpha}_{0,M}(M_k)(x) & = & M_k \\ C^{\alpha}_{1,M}(M_k)(x) & = & 2(1+\alpha)xM_k \\ C^{\alpha}_{2,M}(M_k)(x) & = & [2(2+\alpha)(2k+M+2+2\alpha)x^2 + 2(2+\alpha)(2k+M)]M_k \\ C^{\alpha}_{3,M}(M_k)(x) & = & 4(3+\alpha)(2+\alpha)[(2k+M+2\alpha+4)x^3 + (2k+M+2)x]M_k. \end{array}$$

From these examples and the definition of D_α, it follows immediately that if t is even, the polynomials $C^{\alpha}_{t,M,k}(x)$ only contain even powers of x and if t is odd, the polynomials $C^{\alpha}_{t,M,k}(x)$ only contain odd powers of x.

Now we have the following recursion relation.

Theorem 19 (Recursion formula)**.**

$$C^{\alpha}_{t+1,M}(M_k)(x) = D_\alpha C^{\alpha+1}_{t,M}(M_k)(x).$$

Proof. We immediately calculate that

$$\begin{array}{rcl} C^{\alpha}_{t+1,M}(M_k)(x) & = & D_\alpha D_{\alpha+1}\dots D_{\alpha+t}M_k \\ & = & D_\alpha\left(D_{\alpha+1}\dots D_{\alpha+t}M_k\right) \\ & = & D_\alpha C^{\alpha+1}_{t,M}(M_k)(x). \end{array}$$

□

Clifford-Gegenbauer polynomials of different degree are orthogonal, as is expressed in the following theorem.

Theorem 20 (Orthogonality relation)**.** *If $s \neq t$ then*

$$\langle C^{\alpha}_{s,M}(M_k)(x), C^{\alpha}_{t,M}(M_k)(x)\rangle_\alpha = 0.$$

Proof. Suppose $s > t$. Then

$$\begin{array}{rcl} \langle C^{\alpha}_{s,M}(M_k)(x), C^{\alpha}_{t,M}(M_k)(x)\rangle_\alpha & = & \langle D_\alpha D_{\alpha+1}\dots D_{\alpha+s-1}M_k, C^{\alpha}_{t,M}(M_k)(x)\rangle_\alpha \\ & = & \langle M_k, (\partial_x)^s C^{\alpha}_{t,M}(M_k)(x)\rangle_{\alpha+s} \\ & = & 0, \end{array}$$

by proposition 2 and lemma 4. □

The Clifford-Gegenbauer polynomials also satisfy a partial differential equation in superspace.

Theorem 21 (Differential equation). *$C^{\alpha}_{t,M}(M_k)(x)$ is a solution of the following differential equation:*

$$[(1+x^2)\partial_x^2 + 2(\alpha+1)x\partial_x - C(\alpha,t,M,k)]C^{\alpha}_{t,M}(M_k)(x) = 0$$

with

$$C(\alpha,t,M,k) = \begin{cases} (2\alpha+t+1)(t+M+2k-1), & t \text{ odd} \\ t(2\alpha+t+M+2k), & t \text{ even.} \end{cases}$$

Proof. The theorem can be proved using induction on t. The cases where $t = 0, 1$ are easily checked. We write the following expansion of the Gegenbauer polynomials

$$\begin{array}{lcl} C^{\alpha}_{2t,M}(M_k) & = & \sum_{i=0}^{t} a_{2i}^{2t,\alpha} x^{2i} M_k \\ C^{\alpha}_{2t+1,M}(M_k) & = & \sum_{i=0}^{t} a_{2i+1}^{2t+1,\alpha} x^{2i+1} M_k. \end{array} \tag{4.5}$$

The recursion formula combined with lemma 4 leads to the following relation between the coefficients

$$\begin{array}{lcl} a_{2i}^{2t,\alpha} & = & (2i+2k+M)a_{2i+1}^{2t-1,\alpha+1} + (2\alpha+2i+M+2k)a_{2i-1}^{2t-1,\alpha+1} \\ a_{2i+1}^{2t+1,\alpha} & = & (2i+2)a_{2i+2}^{2t,\alpha+1} + 2(1+\alpha+i)a_{2i}^{2t,\alpha+1}. \end{array}$$

We need to prove the following:

$$\begin{array}{lcl} \partial_x C^{\alpha}_{2t,M}(M_k) & = & 2t(2\alpha+2t+M+2k)C^{\alpha+1}_{2t-1,M}(M_k) \\ \partial_x C^{\alpha}_{2t+1,M}(M_k) & = & (2\alpha+2t+2)(2t+2k+M)C^{\alpha+1}_{2t,M}(M_k) \end{array}$$

or, in terms of the $a_j^{i,\alpha}$:

$$\begin{array}{lcl} 2ia_{2i}^{2t,\alpha} & = & 2t(2\alpha+2t+M+2k)a_{2i-1}^{2t-1,\alpha+1} \\ (2k+2i+M)a_{2i+1}^{2t+1,\alpha} & = & (2\alpha+2t+2)(2t+2k+M)a_{2i}^{2t,\alpha+1}. \end{array}$$

Suppose now that the theorem holds for $C^{\alpha}_{t,M}(M_k)$, $t \leq 2s$. We show that it also holds for $t = 2s+1$. Indeed,

$$\begin{array}{ll} & (2k+2i+M)a_{2i+1}^{2s+1,\alpha} \\ = & (2k+2i+M)((2i+2)a_{2i+2}^{2s,\alpha+1} + (2+2\alpha+2i)a_{2i}^{2s,\alpha+1}) \end{array}$$

$$
\begin{aligned}
&= (2k+2i+M)(2s(2\alpha+2s+M+2k+2)a_{2i+1}^{2s-1,\alpha+2} + (2+2\alpha+2i)a_{2i}^{2s,\alpha+1})\\
&= (2k+2i+M)(2+2\alpha+2i)a_{2i}^{2s,\alpha+1}\\
&\quad +2s(2\alpha+2s+M+2k+2)(a_{2i}^{2s,\alpha+1} - (2\alpha+2i+M+2k+2)a_{2i-1}^{2s-1,\alpha+2})\\
&= (2\alpha+2s+2)(2s+2k+M)a_{2i}^{2s,\alpha+1}\\
&\quad +(2\alpha+2i+M+2k+2)(2ia_{2i}^{2s,\alpha+1} - 2s(2\alpha+2s+M+2k+2)a_{2i-1}^{2s-1,\alpha+2})\\
&= (2\alpha+2s+2)(2s+2k+M)a_{2i}^{2s,\alpha+1}.
\end{aligned}
$$

Similarly we prove that if the theorem holds for $t \leq 2s+1$, then it also holds for $t = 2s+2$. $\square$

Now we can give general formulae for the coefficients of the Clifford-Gegenbauer polynomials, where we use the notations of the previous proof.

Theorem 22 (Explicit form)**.** *If $M \notin -2\mathbb{N}$ and $\alpha \notin -\mathbb{N}$, then the coefficients in the expansion of the Clifford-Gegenbauer polynomials take the following form:*

$$
\begin{aligned}
a_{2i}^{2t,\alpha} &= 2^{2t}\binom{t}{i}\frac{\Gamma(t+k+M/2)}{\Gamma(i+k+M/2)}(\alpha+t+1)_t(\alpha+t+k+M/2)_i\\
a_{2i+1}^{2t+1,\alpha} &= 2^{2t+1}\binom{t}{i}\frac{\Gamma(t+1+k+M/2)}{\Gamma(i+1+k+M/2)}(\alpha+t+1)_{t+1}\\
&\quad \times(\alpha+t+k+M/2+1)_i
\end{aligned}
$$

with $(a)_p = a(a+1)\dots(a+p-1)$ the Pochhammer symbol.

Proof. We first prove the formula for $a_{2i}^{2t,\alpha}$. We obtain, using the expressions from the previous proof:

$$
\begin{aligned}
a_{2i}^{2t,\alpha} &= \frac{t}{i}(2\alpha+2t+M+2k)a_{2i-1}^{2t-1,\alpha+1}\\
&= 4\frac{t}{i}(\alpha+t+M/2+k)(\alpha+t+1)\frac{t+k+M/2-1}{i+k+M/2-1}a_{2i-2}^{2t-2,\alpha+2}\\
&= \dots\\
&= 2^{2i}\binom{t}{i}\frac{(t+k+M/2-1)\dots(t+k-i+M/2)}{(i+k+M/2-1)\dots(k+M/2)}\\
&\quad \times(\alpha+t+1)\dots(\alpha+t+i)\\
&\quad \times(\alpha+t+M/2+k)\dots(\alpha+t+M/2+k+i-1)a_0^{2t-2i,\alpha+2i}\\
&= 2^{2i}\binom{t}{i}\frac{\Gamma(t+k+M/2)}{\Gamma(t+k-i+M/2)}\frac{\Gamma(k+M/2)}{\Gamma(i+k+M/2)}\\
&\quad \times(\alpha+t+1)_i(\alpha+t+k+M/2)_i a_0^{2t-2i,\alpha+2i}.
\end{aligned}
$$

We need a formula for $a_0^{2t,\alpha}$. This can be done as follows:

$$\begin{aligned}
a_0^{2t,\alpha} &= (2k+M)a_1^{2t-1,\alpha+1}\\
&= (2k+M)\frac{2t+M+2k-2}{2k+M}(2\alpha+2t+2)a_0^{2t-2,\alpha+2}\\
&= 4(t+k-1+M/2)(\alpha+t+1)a_0^{2t-2,\alpha+2}\\
&= \ldots\\
&= 2^{2t}(t+k-1+M/2)\ldots(k+M/2)(\alpha+t+1)\ldots(\alpha+2t)a_0^{0,\alpha+2t}\\
&= 2^{2t}\frac{\Gamma(t+k+M/2)}{\Gamma(k+M/2)}(\alpha+t+1)_t.
\end{aligned}$$

Combining these results gives the formula stated in the theorem. The formula for $a_{2i+1}^{2t+1,\alpha}$ follows from

$$a_{2i+1}^{2t+1,\alpha} = (2\alpha+2t+2)\frac{2t+2k+M}{2i+2k+M}a_{2i}^{2t,\alpha+1}.$$

□

As we have that $C^{\alpha}_{t,M}(M_k)(x) = C^{\alpha}_{t,M,k}(x)M_k$ with $C^{\alpha}_{t,M,k}(x)$ a polynomial in the vector variable x, we can compare this polynomial with orthogonal polynomials on the real line. This leads to the following theorem.

Theorem 23. *One has that*

$$\begin{aligned}
C^{\alpha}_{2t,M,k}(x) &= 2^{2t}t!(\alpha+t+1)_t P_t^{\frac{M}{2}+k-1,\alpha}(1+2x^2)\\
C^{\alpha}_{2t+1,M,k}(x) &= 2^{2t+1}t!(\alpha+t+1)_{t+1} x P_t^{\frac{M}{2}+k,\alpha}(1+2x^2),
\end{aligned}$$

where $P_t^{(\alpha,\beta)}$ are the Jacobi polynomials on the real line.

Proof. This follows immediately by comparing the coefficients given in theorem 22 with the definition of the Jacobi polynomials:

$$P_t^{(\alpha,\beta)}(x) = \frac{\Gamma(\alpha+t+1)}{t!\Gamma(\alpha+\beta+t+1)}\sum_{i=0}^{t}\begin{pmatrix} t\\ i\end{pmatrix}\frac{\Gamma(\alpha+\beta+t+i+1)}{\Gamma(\alpha+i+1)}\left(\frac{x-1}{2}\right)^i.$$

□

4.4 A physical application: interpretation of the super-dimension

In basic quantum mechanics, the harmonic oscillator is of the utmost importance. It satisfies the following Schrödinger equation in $\mathbb{R}^m$

$$\frac{1}{2}\left(\Delta_b - \underline{x}^2\right)\phi = E\phi,$$

using units $\hbar = m = \omega = 1$. A canonical extension of this model to superspace would thus be

$$\frac{1}{2}\left(\Delta - x^2\right)\phi = E\phi,$$

where we have replaced the Laplace operator and the vector variable by their super analogues. A direct calculation now shows that every function ϕ of the form $\phi = \exp(x^2/2)H_{t,M}(M_k)$ with $M_k \in \mathcal{M}_k$ is a solution of this equation with corresponding energy $E = M/2 + (t+k)$. Furthermore, for a given energy $E_T = M/2 + T$, there are exactly

$$\begin{aligned}
\sum_{i=0}^{T} \dim \mathcal{M}_i &= \sum_{i=0}^{T} \sum_{j=0}^{\min(i,2n)} \binom{2n}{j}\binom{i-j+m-2}{m-2} \\
&= \sum_{i=0}^{\min(T,2n)} \binom{2n}{i}\binom{T-i+m-1}{m-1}
\end{aligned}$$

eigenfunctions. The second equality follows from the Fischer decomposition or by a direct calculation (see [32]).

Moreover this is in correspondence with what would be expected physically. Indeed, the number of eigenfunctions with energy E_T is the total number of possibilities for selecting T particles out of a set of m bosonic and $2n$ fermionic particles.

This can also be seen in the following way. If we put

$$\begin{aligned}
a_i^+ &= \tfrac{\sqrt{2}}{2}(x_i - \partial_{x_i}) & a_i^- &= \tfrac{\sqrt{2}}{2}(x_i + \partial_{x_i}) \\
b_{2i}^+ &= \tfrac{1}{2}(x\grave{}_{2i} + 2\partial_{x\grave{}_{2i-1}}) & b_{2i}^- &= \tfrac{1}{2}(x\grave{}_{2i-1} + 2\partial_{x\grave{}_{2i}}) \\
b_{2i-1}^+ &= \tfrac{1}{2}(x\grave{}_{2i-1} - 2\partial_{x\grave{}_{2i}}) & b_{2i-1}^- &= \tfrac{1}{2}(-x\grave{}_{2i} + 2\partial_{x\grave{}_{2i-1}})
\end{aligned}$$

we can rewrite the Hamiltonian as

$$H = \frac{1}{2}(\Delta - x^2) = \sum_{i=1}^{m} a_i^+ a_i^- + \sum_{i=1}^{2n} b_i^+ b_i^- + \frac{M}{2}.$$

As the operators $a_i^\pm$, $b_i^\pm$ satisfy

$$\begin{array}{ll} \left[a_i^\pm, a_j^\pm\right] = 0 & \{b_i^\pm, b_j^\pm\} = 0 \\ \left[a_i^-, a_j^+\right] = \delta_{ij} & \{b_i^+, b_j^-\} = \delta_{ij} \\ \left[a_i^\pm, b_j^\pm\right] = 0 & \left[a_i^\mp, b_j^\pm\right] = 0, \end{array}$$

this is the canonical realization of an oscillator with m bosonic and $2n$ fermionic degrees of freedom. The ground level energy is given by $M/2$, which gives us a physical interpretation of the super-dimension. The reader should compare this approach with e.g. the one given in [52] for the purely fermionic case.

Finally, the Hamiltonian can also be factorized in toto using Clifford numbers. Indeed, putting

$$Q_+ = \tfrac{1}{2}(\partial_x + x) \qquad Q_- = \tfrac{1}{2}(\partial_x - x)$$

we have that

$$H = \{Q_+, Q_-\}.$$

Remark 15. *In chapter 9 we will discuss some problems related to more general Schrödinger equations in superspace.*

Chapter 5

Integration in superspace

In this chapter we will discuss integration in superspace. Usually, the so-called Berezin integral (see e.g. [8, 73]) is used, but without sound mathematical justification. We will show how this integral naturally arises from the theory of harmonic analysis in superspace. First of all, we will discuss the formula of Pizzetti in $\mathbb{R}^m$. This formula originally motivated us to give a definition of integration over the supersphere in [31]. In the next section we will establish a set of properties that uniquely determine the Pizzetti integral in superspace. We will then prove some important properties of this integral, such as a Funk-Hecke theorem etc. In the final section we will use a generalized form of integration in spherical co-ordinates to obtain an integral over the whole superspace and we will prove that this integral is equivalent with the Berezin integral.

Note that also other attempts have been made to connect the Berezin integral with more familiar types of integration, such as given in [81] and [83] using contour integrals.

The results of this chapter have been either published (see [31]) or submitted for publication (see [26]), although in this chapter the presentation has been made more logical.

5.1 Pizzetti's formula in $\mathbb{R}^m$

Although not very well known, there exist explicit and easy formulae to calculate the integral of an arbitrary polynomial over the unit-sphere in $\mathbb{R}^m$, see for example the recent papers [3, 53]. However, it is not obvious how to extend

these formulae in a consistent manner to superspace. To that end we will draw inspiration from an old result of Pizzetti, see [79], expressing the integration over the sphere as an infinite sum of powers of the Laplace operator (see formula (5.1)). First we will show how this formula is obtained.
We are looking for a formula to calculate

$$\int_{\mathbb{S}^{m-1}} R \, d\sigma$$

with R an arbitrary polynomial, $\mathbb{S}^{m-1}$ the unit-sphere in $\mathbb{R}^m$ and $d\sigma$ the classical Lebesgue surface measure.
We have to consider two cases:
1) $R = R_{2k}$, a homogeneous polynomial of even degree $2k$
We then calculate

$$\begin{aligned}
\int_{\mathbb{S}^{m-1}} R_{2k} \, d\sigma &= \int_{\mathbb{S}^{m-1}} \sum_{i=0}^{k} \underline{x}^{2i} H_{2k-2i} \, d\sigma \\
&= \sum_{i=0}^{k} \int_{\mathbb{S}^{m-1}} \underline{x}^{2i} H_{2k-2i} \, d\sigma \\
&= \sum_{i=0}^{k} (-1)^i \int_{\mathbb{S}^{m-1}} H_{2k-2i} \, d\sigma \\
&= (-1)^k \int_{\mathbb{S}^{m-1}} H_0 \, d\sigma \\
&= (-1)^k H_0 \frac{2\pi^{m/2}}{\Gamma(m/2)}.
\end{aligned}$$

In this calculation we have used the Fischer-decomposition and the orthogonality of spherical harmonics of different degree on $\mathbb{S}^{m-1}$. We can determine H_0 by using the projection operator $\mathbb{P}_k^{2k}$ (see theorem 5) where $M = m$. This gives

$$H_0 = \frac{1}{2^{2k} k!} \frac{\Gamma(m/2)}{\Gamma(k + m/2)} \Delta_b^k R_{2k}.$$

We conclude that

$$\int_{\mathbb{S}^{m-1}} R_{2k} \, d\sigma = (-1)^k \frac{2\pi^{m/2}}{2^{2k} k! \Gamma(k + m/2)} \Delta_b^k R_{2k}.$$

2) $R = R_{2k+1}$, a homogeneous polynomial of odd degree $(2k + 1)$

We then have that

$$\int_{\mathbb{S}^{m-1}} R_{2k+1}\, d\sigma = 0$$

using the same reasoning or using a symmetry argument.

Both results can be summarized in one formula. So let R be an arbitrary polynomial, then

$$\int_{\mathbb{S}^{m-1}} R\, d\sigma = \sum_{k=0}^{\infty} (-1)^k \frac{2\pi^{m/2}}{2^{2k} k! \Gamma(k + m/2)} (\Delta_b^k R)(0) \tag{5.1}$$

where the right-hand side has to be evaluated in the origin of $\mathbb{R}^m$. This formula is the so-called Pizzetti formula. It now seems logical to generalize this formula to superspace as follows:

$$\int_{SS} R = \sum_{k=0}^{\infty} (-1)^k \frac{2\pi^{M/2}}{2^{2k} k! \Gamma(k + M/2)} (\Delta^k R)(0)$$

by substituting M for m and Δ for Δ_b and to use this formula as a definition of an integral over the supersphere. This turns out to be a good choice, as we will show in the following section.

Remark 16. *The formula of Pizzetti can be rewritten in a more elegant form using Bessel functions, as follows:*

$$\int_{\mathbb{S}^{m-1}} R\, d\sigma = 2\pi^{m/2} (P_{\frac{m}{2}-1}(\partial_{\underline{x}}) R)(0)$$

with

$$P_{\frac{m}{2}-1}(z) = (z/2)^{1-\frac{m}{2}} J_{\frac{m}{2}-1}(z)$$

a kind of 'normalized' Bessel function, and $\partial_{\underline{x}}$ the Dirac-operator.

5.2 Integration over the supersphere

5.2.1 The problem of integration over the supersphere

In this and the following sections we restrict ourselves to the case $M = m - 2n \notin -2\mathbb{N}$. This assumption allows us to use the Fischer decomposition (theorem 4) and the corresponding projections (see theorem 5).

We start by putting forward a set of properties we want an integral over the supersphere to exhibit. The supersphere is the formal object defined by the algebraic equation $x^2 = -1$ (if $n = 0$ this is exactly the equation of the unit-sphere in $\mathbb{R}^m$).

Definition 8. *A linear functional $T : \mathcal{P} \to \mathbb{R}$ is called an integration over the supersphere if it satisfies the following properties for all $f(x) \in \mathcal{P}$:*

1. $T(x^2 f(x)) = -T(f(x))$
2. $T(f(g \cdot x)) = T(f(x)), \quad \forall g \in SO(m) \times Sp(2n)$.

These two properties are of course very natural. The first one says that we can work modulo $x^2 + 1$ (the equation of the supersphere). The second property is just the invariance of integrals under the action of $SO(m) \times Sp(2n)$ (generalizing the rotational invariance of integration over the sphere in $\mathbb{R}^m$).

We will now determine the set of all functionals satisfying these two properties. More precisely, we will obtain the following theorem.

Theorem 24. *The space of all linear functionals $T : \mathcal{P} \to \mathbb{R}$ satisfying the properties 1 and 2 of definition 8 is a finite-dimensional vectorspace of dimension $n + 1$.*

In section 5.2.3 we will determine a way to distinguish between these different types of integration.

We first prove the following lemma, which will show to be crucial to the further development of this section.

Lemma 12. *Let $\mathcal{H}$ be a subspace of $\mathcal{P}$ of dimension $\dim \mathcal{H} > 1$, irreducible under the action of $SO(m) \times Sp(2n)$. Then an integral T over the supersphere of a function $f \in \mathcal{H}$ is always zero.*

Proof. If $T : \mathcal{H} \to \mathbb{R}$ is a linear functional satisfying the requirements of definition 8, then we have that $\ker T$ is invariant under the action of $SO(m) \times Sp(2n)$. As $\ker T$ is a subspace of $\mathcal{H}$, we have that either $\ker T = 0$ or $\ker T = \mathcal{H}$, due to the irreducibility of $\mathcal{H}$. As moreover $\dim(\ker T) > 0$ we have that $\ker T = \mathcal{H}$ and the lemma follows. (Note that this is just an application of Schur's lemma.) $\square$

Now we will determine all possible integrations over $\mathcal{P}$. Recall that we have the following Fischer decomposition

$$\mathcal{P} = \bigoplus_{j=0}^{\infty} \bigoplus_{k=0}^{\infty} x^{2j} \mathcal{H}_k.$$

We thus have for a general integral T that

$$T(\mathcal{P}) = \sum_{j=0}^{\infty}\sum_{k=0}^{\infty}(-1)^j T(\mathcal{H}_k).$$

Hence it suffices to know how integrations on $\mathcal{H}_k$ look like. In theorem 8 we have found that $\mathcal{H}_k$ decomposes into irreducible pieces as

$$\mathcal{H}_k = \bigoplus_{i=0}^{\min(n,k)} \mathcal{H}^b_{k-i}\otimes\mathcal{H}^f_i \;\oplus\; \bigoplus_{j=0}^{\min(n,k-1)-1}\;\bigoplus_{l=1}^{\min(n-j,\lfloor\frac{k-j}{2}\rfloor)} f_{l,k-2l-j,j}\mathcal{H}^b_{k-2l-j}\otimes\mathcal{H}^f_j. \tag{5.2}$$

As a consequence of lemma 12, we see that only the one-dimensional pieces in the decomposition of $\mathcal{H}_k$ can give rise to an integration which is not zero. From formula (5.2) we conclude that there are exactly $n+1$ of these pieces, namely

$$f_{i,0,0}\,\mathcal{H}^b_0\otimes\mathcal{H}^f_0 \;\;\subset\;\; \mathcal{H}_{2i}, \quad i=0,\ldots,n.$$

On each of these pieces the value of the integral T can be freely chosen as there are no further restrictions. We denote these chosen values by $a_i \in \mathbb{R}$, $i=0,\ldots,n$.

Hence we have reduced the problem under consideration to constructing projections of elements of $\mathcal{P}$ on these one-dimensional irreducible pieces. The general form of an integration over the supersphere, satisfying definition 8, can then be found as follows.

Suppose we are given a polynomial $R\in\mathcal{P}$, then we perform the following projections

$$R \xrightarrow{\quad\mathbb{P}_{2k}\quad} \mathcal{P}_{2k} \xrightarrow{\quad x^{2k-2i}\mathbb{P}^{2k}_{k-i}\quad} x^{2k-2i}\mathcal{H}_{2i} \xrightarrow{\quad\mathbb{P}^{bf}_{2i}\quad} f_{i,0,0}\mathcal{H}^b_0\otimes\mathcal{H}^f_0$$

with

- $\mathbb{P}_{2k}$ the projection onto the space of homogeneous polynomials of degree $2k$
- $\mathbb{P}^{2k}_{k-i}$ the projection from the space of homogeneous polynomials of degree $2k$ to the space of spherical harmonics of degree $2i$ (Fischer decomposition)
- $\mathbb{P}^{bf}_{2i}$ the projection from the space of spherical harmonics of degree $2i$ to its unique one-dimensional irreducible subspace $f_{i,0,0}\mathcal{H}^b_0\otimes\mathcal{H}^f_0$.

These projections have to be performed for all values of k and for $i = 0, \ldots, n$. Moreover, we only have to consider projections on these pieces, as all the other components of R are elements of irreducible subspaces of dimension larger than one and hence, by lemma 12, do not contribute to the integral.

Summarizing, we arrive at the following general form for an integral over the supersphere:

$$T = \sum_{i=0}^{n} \frac{a_i}{f_{i,0,0}} \sum_{k=i}^{\infty} (-1)^{k-i} \mathbb{P}_{2i}^{bf} \mathbb{P}_{k-i}^{2k} \mathbb{P}_{2k}. \tag{5.3}$$

The factor $(-1)^{k-i}$ stems from the fact that $x^{2k-2i}\mathcal{H}_{2i}$ equals $(-1)^{k-i}\mathcal{H}_{2i}$ on the supersphere $x^2 = -1$.

As there are exactly $n+1$ one-dimensional irreducible subspaces $f_{i,0,0}\ \mathcal{H}_0^b \otimes \mathcal{H}_0^f$, and thus $n+1$ values a_i to be chosen, this proves theorem 24.

Let us now find explicit formulae for the projection operators. The operators $\mathbb{P}_{k-i}^{2k}$ follow from the Fischer decomposition (see theorem 5). Next we construct the operator $\mathbb{P}_{2i}^{bf}$. Of course we have that $\mathbb{P}_{2i}^{bf} = \mathbb{Q}_{i,0}^{2i}$ (see theorem 8) but we can obtain a simpler expression for this projection operator. First recall that this operator is the projection

$$\mathbb{P}_{2i}^{bf} : \mathcal{H}_{2i} \longrightarrow f_{i,0,0}\mathcal{H}_0^b \otimes \mathcal{H}_0^f.$$

It is immediately clear that Δ_b^i annihilates all terms in the decomposition of $\mathcal{H}_{2i}$ except for the term $f_{i,0,0}\mathcal{H}_0^b \otimes \mathcal{H}_0^f$. Indeed, we have $(i > 0)$

$$\begin{aligned}
\Delta_b^i(f_{i,0,0}\mathcal{H}_0^b \otimes \mathcal{H}_0^f) &= \Delta_b^i(f_{i,0,0})\mathcal{H}_0^b \otimes \mathcal{H}_0^f \\
&= \frac{n!}{\Gamma(\frac{m}{2}+i)} \Delta_b^i(\underline{x}^{2i})\mathcal{H}_0^b \otimes \mathcal{H}_0^f \\
&= \frac{n!}{\Gamma(\frac{m}{2}+i)} 2^{2i} i! \frac{\Gamma(\frac{m}{2}+i)}{\Gamma(\frac{m}{2})} \mathcal{H}_0^b \otimes \mathcal{H}_0^f \\
&= \frac{n! i! 2^{2i}}{\Gamma(\frac{m}{2})} \mathcal{H}_0^b \otimes \mathcal{H}_0^f.
\end{aligned}$$

So we conclude that the operator $\mathbb{P}_{2i}^{bf}$ takes the following form

$$\mathbb{P}_{2i}^{bf} = \frac{\Gamma(\frac{m}{2})}{n! i! 2^{2i}} f_{i,0,0} \Delta_b^i.$$

Note that in the case where $i = 0$ no projection is necessary ($\mathbb{P}_0^{bf} = 1$) since $\mathcal{H}_0$ is already one-dimensional.

Finally, a general integral T over the supersphere can also be written as

$$T = \sum_{i=0}^{n} a_i \int_i$$

with

$$\int_i = \sum_{k=i}^{\infty} \frac{(-1)^{k-i}}{f_{i,0,0}} \mathbb{P}_{2i}^{bf} \mathbb{P}_{k-i}^{2k} \mathbb{P}_{2k}$$

such that

$$\int_i f_{j,0,0} = \delta_{ij}.$$

In the next subsection we will construct some explicit examples.

5.2.2 Some examples

The Pizzetti case

This integral is defined by putting $a_0 = 2\pi^{M/2}/\Gamma(M/2)$ and $a_i = 0$, $i > 0$. When explicitly writing the proper projections we arrive at the following definition.

Definition 9 (Pizzetti)**.** *The integral of $R \in \mathcal{P}$ over the supersphere is given by*

$$\int_{SS} R = \sum_{k=0}^{\infty} (-1)^k \frac{2\pi^{M/2}}{4^k k! \Gamma(k + M/2)} (\Delta^k R)(0), \tag{5.4}$$

where $(\Delta^k R)(0)$ means evaluating the result in $x_i = \grave{x}_i = 0$.

The normalization is chosen such that $\int_{SS} 1$ gives the area of the sphere in the purely bosonic case. This is the same formula as was proven by Pizzetti in $\mathbb{R}^m$, see section 5.1.

Combining this formula with the concept of integration in spherical coordinates will yield the so-called Berezin integral (see section 5.3).

Another possibility

We consider the simplest case (with exception of the Pizzetti integral). This is the integral where $a_1 \neq 0$ but $a_i = 0$, $i \neq 1$. Explicitly determining the projections in formula (5.3) leads to

Definition 10. *The integral of a superpolynomial R over the supersphere is given by*

$$\int_{SS,1} R = c \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{4^{k+1}(k-1)!\Gamma(k+M/2+1)} \left(\Delta_b(2M\Delta^{k-1} - x^2\Delta^k)R\right)(0). \tag{5.5}$$

This can be simplified further to

$$\int_{SS,1} R = c \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{4^{k+1}(k-1)!\Gamma(k+M/2+1)} \left(2M\Delta_b\Delta^{k-1} - 2m\Delta^k)R\right)(0).$$

In this definition c is a constant which still has to be determined. If we take

$$c = \frac{\Gamma(2+M/2)\Gamma(1+m/2)}{mMn!}$$

then the integral is normalized such that

$$\int_{SS,1} f_{1,0,0} = 1.$$

Note that this second possibility of integration over the supersphere is already a lot more complicated than the Pizzetti integral. In a similar way we can of course construct the integral corresponding to the general case $a_j \neq 0$, $a_i = 0, i \neq j$.

5.2.3 Distinction between the different types of integration

In the purely bosonic case (where $n = 0$), there is only one possibility for integration over the sphere, namely Pizzetti's formula, because there is only one one-dimensional space $\mathcal{H}_0^b$. In this section we will show how this particular possibility can be distinguished from the other ones.

We start with the following definition of orthogonality:

Definition 11. *The space $\mathcal{H}_k$ is orthogonal to $\mathcal{H}_l$ ($k \neq l$), notation $\mathcal{H}_k \perp \mathcal{H}_l$, with respect to the integral T over the supersphere if and only if*

$$T(\mathcal{H}_k\mathcal{H}_l) = 0 = T(\mathcal{H}_l\mathcal{H}_k).$$

Let us now first prove the following technical lemma.

Lemma 13. *Let H_k, H_l be spherical harmonics of degree k, l, i.e. $H_k \in \mathcal{H}_k$ and $H_l \in \mathcal{H}_l$. Then one has that*

$$\Delta(H_k H_l) = \sum H_{k-1} H_{l-1}$$

where the right-hand side is a sum of products of spherical harmonics of degree $k-1$ and $l-1$.

Proof. First note that we can split $H_k = H_k^+ + H_k^-$ where the terms in H_k^+ contain only even numbers of anti-commuting co-ordinates and H_k^- only odd numbers. Moreover $\Delta H_k^+ = 0 = \Delta H_k^-$, because Δ is an even operator.
We give the proof for $H_k^+ H_l$, the other part being similar. We have that

$$\partial_{x_i}^2 (H_k^+ H_l) = \partial_{x_i}^2 (H_k^+) H_l + 2\partial_{x_i}(H_k^+)\partial_{x_i}(H_l) + H_k^+ \partial_{x_i}^2 (H_l)$$

and

$$\begin{aligned}
\partial_{\grave{x}_{2j-1}} \partial_{\grave{x}_{2j}} (H_k^+ H_l) &= \partial_{\grave{x}_{2j-1}} \partial_{\grave{x}_{2j}} (H_k^+) H_l - \partial_{\grave{x}_{2j}} (H_k^+) \partial_{\grave{x}_{2j-1}} (H_l) \\
&\quad + \partial_{\grave{x}_{2j-1}} (H_k^+) \partial_{\grave{x}_{2j}} (H_l) + H_k^+ \partial_{\grave{x}_{2j-1}} \partial_{\grave{x}_{2j}} (H_l).
\end{aligned}$$

So we obtain

$$\begin{aligned}
\Delta(H_k^+ H_l) &= \Delta(H_k^+) H_l + H_k^+ \Delta(H_l) + \sum H_{k-1} H_{l-1} \\
&= \sum H_{k-1} H_{l-1}
\end{aligned}$$

due to the harmonicity of H_k and H_l. □

We then have the following orthogonality property of the Pizzetti integral.

Theorem 25 (Orthogonality)**.** *Let H_k, H_l be spherical harmonics of degree k, l. If $k \neq l$ then*

$$\int_{SS} H_k H_l = 0 = \int_{SS} H_l H_k.$$

Proof. The case where $k + l$ is odd is trivial, because formula (5.4) gives zero for homogeneous polynomials of odd degree. Now let us assume that $k + l$ is even, say $k + l = 2p$. Then the integral reduces to

$$\int_{SS} H_k H_l = \frac{2\pi^{M/2}}{2^{2p} p! \Gamma(p + M/2)} \Delta^p (H_k H_l).$$

Now we can apply lemma 13. Suppose e.g. that $k < l$. Then applying lemma 13 k times yields:

$$\Delta^k(H_k H_l) = \sum H_0 H_{l-k}.$$

One more action of Δ gives zero, because all the factors H_0 are constants. So if $k \neq l$ then

$$\int_{SS} H_k H_l = 0 = \int_{SS} H_l H_k.$$

□

We can prove that the Pizzetti integral is the only integral that satisfies this orthogonality property.

Theorem 26. *The Pizzetti integral over the supersphere is the only integral that has the property*

$$k \neq l \quad \Longrightarrow \quad \mathcal{H}_k \perp \mathcal{H}_l. \tag{5.6}$$

Proof. The fact that the Pizzetti integral satisfies (5.6) is proven in theorem 25. Conversely, a general integral on the supersphere has the following form:

$$\int = \sum_{i=0}^{n} \frac{a_i}{f_{i,0,0}} \sum_{k=i}^{\infty} (-1)^{k-i} \mathbb{P}_{2i}^{bf} \mathbb{P}_{k-i}^{2k} \mathbb{P}_{2k}.$$

If it is not the Pizzetti integral, then there exists a maximal $t > 0$ such that $a_t \neq 0$. Then one has

$$\int f_{t,0,0} = a_t \neq 0$$

which immediately implies that $\mathcal{H}_0$ is not orthogonal to $\mathcal{H}_{2t}$. □

One can even go one step further. Not only the spaces $\mathcal{H}_k$ are mutually orthogonal with respect to the Pizzetti integral, but in fact all irreducible pieces (see theorem 8) are mutually orthogonal. This is summarized in the following theorem. The proof is essentially a reduction to either the purely bosonic or the purely fermionic case.

Theorem 27. *One has that*

$$f_{i,p,q}\mathcal{H}_p^b \otimes \mathcal{H}_q^f \quad \perp \quad f_{j,r,s}\mathcal{H}_r^b \otimes \mathcal{H}_s^f$$

with respect to the Pizzetti integral if and only if $(i,p,q) \neq (j,r,s)$.

Proof. It is only necessary to prove this statement for the irreducible pieces contained in the same $\mathcal{H}_k$. So we prove that

$$f_{i,k-2i-p,p}\mathcal{H}^b_{k-2i-p} \otimes \mathcal{H}^f_p \quad \perp \quad f_{j,k-2j-q,q}\mathcal{H}^b_{k-2j-q} \otimes \mathcal{H}^f_q$$

with either $p \neq q$ or $p = q$, $i \neq j$. Due to the definition of the Pizzetti integral (formula (5.4)) it suffices to prove that

$$\Delta^k(f_{i,k-2i-p,p}\mathcal{H}^b_{k-2i-p} \otimes \mathcal{H}^f_p \; f_{j,k-2j-q,q}\mathcal{H}^b_{k-2j-q} \otimes \mathcal{H}^f_q) = 0.$$

As we also have

$$\begin{aligned} f_{i,k-2i-p,p} &= \sum_{s=0}^{i} a_s \underline{x}^{2s} \underline{\grave{x}}^{2i-2s} \\ f_{j,k-2j-q,q} &= \sum_{t=0}^{j} b_t \underline{x}^{2t} \underline{\grave{x}}^{2j-2t}, \end{aligned}$$

it is moreover sufficient to consider a term of the form

$$\begin{aligned} &\Delta^k \left(\underline{x}^{2s} \underline{\grave{x}}^{2i-2s} \mathcal{H}^b_{k-2i-p} \otimes \mathcal{H}^f_p \underline{x}^{2t} \underline{\grave{x}}^{2j-2t} \mathcal{H}^b_{k-2j-q} \otimes \mathcal{H}^f_q \right) \\ &= \Delta^k \left(\underline{x}^{2s+2t} \mathcal{H}^b_{k-2i-p} \mathcal{H}^b_{k-2j-q} \underline{\grave{x}}^{2i+2j-2s-2t} \mathcal{H}^f_p \mathcal{H}^f_q \right). \end{aligned}$$

Note that in this equation Δ^k can be expanded as

$$\Delta^k = \sum_{u=0}^{k} c_u \Delta_b^{k-u} \Delta_f^u.$$

If $p+q$ is odd, then all terms vanish; if $p+q$ is even there remains exactly one term, namely where $2u = 2i + 2j - 2s - 2t + p + q$. We obtain

$$\begin{aligned} &\Delta_b^{k-u} \left(\underline{x}^{2s+2t} \mathcal{H}^b_{k-2i-p} \mathcal{H}^b_{k-2j-q} \right) \Delta_f^u \left(\underline{\grave{x}}^{2i+2j-2s-2t} \mathcal{H}^f_p \mathcal{H}^f_q \right) \\ &= \text{constant} \times \Delta_b^{k-i-j-\frac{p+q}{2}} \left(\mathcal{H}^b_{k-2i-p} \mathcal{H}^b_{k-2j-q} \right) \Delta_f^{\frac{p+q}{2}} \left(\mathcal{H}^f_p \mathcal{H}^f_q \right), \end{aligned}$$

where we have used lemma 2 in the second line.

If $p \neq q$, the second term is always zero (apply lemma 13 as in the proof of theorem 25); if $p = q$ then $i \neq j$ and the first term is always zero (again applying lemma 13). This shows that both spaces are indeed orthogonal. □

Finally, let us summarize the results concerning integration over the supersphere in the following theorem.

Theorem 28. *If $M \notin -2\mathbb{N}$, the only linear functional $T : \mathcal{P} \rightarrow \mathbb{R}$ satisfying the following properties for all $f(x) \in \mathcal{P}$:*

- $T(x^2 f(x)) = -T(f(x))$
- $T(f(g \cdot x)) = T(f(x)), \quad \forall g \in SO(m) \times Sp(2n)$
- $k \neq l \quad \Longrightarrow \quad T(\mathcal{H}_k \mathcal{H}_l) = 0 = T(\mathcal{H}_l \mathcal{H}_k)$
- $T(1) = \dfrac{2\pi^{M/2}}{\Gamma(M/2)}$,

is given by the Pizzetti integral

$$\int_{SS} R = \sum_{k=0}^{\infty} (-1)^k \frac{2\pi^{M/2}}{4^k k! \Gamma(k + M/2)} (\Delta^k R)(0).$$

In the purely fermionic case $m = 0$ we can use the same techniques as before. Indeed, we still have a Fischer decomposition and projection operators available. The only difference is that we need to normalize the integral in a different way, because in this case we have

$$T(1) = \frac{2\pi^{M/2}}{\Gamma(M/2)} = 0.$$

So normalizing to $T(1) = 1$, we obtain in the purely fermionic case

Theorem 29. *The only linear functional $T : \Lambda_{2n} \rightarrow \mathbb{R}$ satisfying the following properties for all $f(x) \in \Lambda_{2n}$:*

- $T(\underline{x}^2 f(x)) = -T(f(x))$
- $T(f(g \cdot x)) = T(f(x)), \quad \forall g \in Sp(2n)$
- $k \neq l \quad \Longrightarrow \quad T(\mathcal{H}_k^f \mathcal{H}_l^f) = 0 = T(\mathcal{H}_l^f \mathcal{H}_k^f)$
- $T(1) = 1$

is given by the Pizzetti integral

$$\int_{SS} R = \sum_{k=0}^{n} \frac{(n-k)!}{4^k k! n!} (\Delta_f^k R)(0).$$

Note that in this case the summation ends after n terms because higher powers of the fermionic Laplace operator are identically zero.

5.2.4 Extension and basic properties

In the previous sections we have restricted ourselves to the case $M \notin -2\mathbb{N}$ or $m = 0$. We now extend our definition to the case $M \in -2\mathbb{N}$ and define the integral over the supersphere as follows.

Definition 12. *The integral of a superpolynomial $R \in \mathcal{P}$ over the supersphere is given by*

$$\int_{SS} R = \sum_{k=0}^{\infty} (-1)^k \frac{2\pi^{M/2}}{2^{2k} k! \Gamma(k + M/2)} (\Delta^k R)(0). \tag{5.7}$$

If $m = 0$, the integral is given by

$$\int_{SS} R = \sum_{k=0}^{n} \frac{(n-k)!}{4^k k! n!} (\Delta_f^k R)(0). \tag{5.8}$$

Note that in the case where $M = -2t$ ($m \neq 0$), the first t terms in the summation vanish and formula (5.7) reduces to:

$$\int_{SS, M=-2t} R = \sum_{k=t+1}^{\infty} (-1)^k \frac{2\pi^{M/2}}{2^{2k} k! \Gamma(k + M/2)} (\Delta^k R)(0).$$

Furthermore we define the area of the supersphere by

$$\sigma_M = \int_{SS} 1 = \frac{2\pi^{M/2}}{\Gamma(M/2)}.$$

Remark 17. *The area of the supersphere should not be interpreted in a set-theoretical way. Indeed,*

- *in case $M = -2t - 1$ the area of the supersphere can be negative (compare with the graph of the Gamma-function)*
- *in case $M = -2t$ the area of the supersphere is zero.*

Note that even in the case where $M \in -2\mathbb{N}$ we still have, using lemma 2, that

$$\int_{SS} x^2 f = - \int_{SS} f.$$

Also spaces of spherical harmonics of different degree remain orthogonal. The only difference is that we cannot prove that formula (5.7) is the unique formula satisfying these properties, because we do not have a Fischer decomposition at our disposal.

As an immediate consequence of definition 12 we have the following proposition.

Proposition 3 (Mean value)**.** *Let f be a harmonic polynomial. Then*

$$\int_{SS} f = \frac{2\pi^{M/2}}{\Gamma(M/2)} f(0).$$

5.2.5 The superball and Green's theorem

In a similar way as for the supersphere, we can define an integral over the superball using the analogy with the Pizzetti formula. Indeed, it is easy to check that the integral of a polynomial R over the unit-ball $B(0,1)$ in $\mathbb{R}^m$ is given by

$$\int_{B(0,1)} R\, dV(\underline{x}) = \sum_{k=0}^{\infty} (-1)^k \frac{\pi^{m/2}}{2^{2k} k! \Gamma(k + m/2 + 1)} (\Delta_b^k R)(0).$$

If we make the substitutions $m \rightarrow M$ and $\Delta_b \rightarrow \Delta$, we obtain the following definition for integration over the superball.

Definition 13. *The integral of a polynomial $R \in \mathcal{P}$ over the superball is given by*

$$\int_{SB} R = \sum_{k=0}^{\infty} (-1)^k \frac{\pi^{M/2}}{2^{2k} k! \Gamma(k + M/2 + 1)} (\Delta^k R)(0).$$

We will need the following technical result.

Lemma 14. *Let $R_{2t+1} \in \mathcal{P}_{2t+1} \otimes \mathcal{C}$. Then one has*

$$\Delta^{t+1}(x R_{2t+1}) = 2(t+1)\Delta^t \partial_x (R_{2t+1}).$$

Proof. We know that (see theorem 3)

$$\Delta(x R_{2t+1}) = 2\partial_x R_{2t+1} + x\Delta R_{2t+1}$$

and so

$$\begin{aligned}
\Delta^{t+1}(xR_{2t+1}) &= \Delta^t(2\partial_x R_{2t+1} + x\Delta R_{2t+1}) \\
&= (\sum_{i=0}^{t+1} 2)\, \partial_x^{2t+1}(R_{2t+1}) \\
&= 2(t+1)\partial_x^{2t+1}(R_{2t+1}).
\end{aligned}$$

□

We can now easily prove the following generalization of Green's theorem.

Theorem 30 (Green I)**.** *Let $R \in \mathcal{P} \otimes \mathcal{C}$ be a $\mathcal{C}$-valued superpolynomial. Then one has*

$$\begin{array}{llll}
(i) & \int_{SS} xR & = & -\int_{SB} \partial_x R \\
(ii) & \int_{SS} \Gamma(R) & = & 0 \\
(iii) & \int_{SS} \mathbb{E}R & = & -\int_{SB} \Delta R.
\end{array}$$

Proof. For the first formula, we only need to prove the case where $R = R_{2t+1} \in \mathcal{P}_{2t+1}$. In that case we have, using lemma 14,

$$\begin{aligned}
\int_{SS} xR_{2t+1} &= (-1)^{t+1} \frac{2\pi^{M/2}}{2^{2t+2}(t+1)!\Gamma(t+M/2+1)} (\Delta^{t+1} xR_{2t+1})(0) \\
&= (-1)^{t+1} \frac{2\pi^{M/2} 2(t+1)}{2^{2t+2}(t+1)!\Gamma(t+M/2+1)} (\Delta^t \partial_x R_{2t+1})(0)
\end{aligned}$$

and on the other hand

$$\int_{SB} \partial_x R_{2t+1} = (-1)^t \frac{\pi^{M/2}}{2^{2t} t! \Gamma(t+M/2+1)} \Delta^t \partial_x R_{2t+1}.$$

The second formula is trivial, because $[\Gamma, \Delta] = 0$.
The third formula is found either by combining the previous statements using $\mathbb{E} = x\partial_x - \Gamma$, or again by direct calculation. □

As a consequence we immediately have the following

Corollary 3. *Let $R \in \mathcal{P}$ be a superpolynomial. Then*

$$\int_{SS} \Delta_{LB}(R) = 0.$$

We also have the following theorem, which is classically used to prove the orthogonality of spherical harmonics of different degree. We were only able to prove this theorem under the assumption that $M \notin -2\mathbb{N}$, as the proof makes extensive use of the Fischer decomposition.

Theorem 31 (Green II)**.** *Let $f, g \in \mathcal{P}$ be two superpolynomials and let $M \notin -2\mathbb{N}$. Then*

$$\int_{SS} (f\mathbb{E}g - (\mathbb{E}f)g) = -\int_{SB} (f\Delta g - (\Delta f)g).$$

Proof. As $M \notin -2\mathbb{N}$, the Fischer-decomposition exists, so it suffices to consider functions of the following form

$$\begin{array}{rcll} f & = & x^{2k}H, & \quad H \in \mathcal{H}_i \\ g & = & x^{2l}K, & \quad K \in \mathcal{H}_j. \end{array}$$

We distinguish two cases.

1) The case where $i \neq j$. Then the left-hand side reduces to

$$\int_{SS} (f\mathbb{E}g - (\mathbb{E}f)g) = (2l + j - 2k - i) \int_{SS} x^{2k+2l} HK = 0$$

using the orthonality of spherical harmonics of different degree. So we need to prove that also the right-hand side is zero. We have that

$$\begin{array}{rcl} f\Delta g - (\Delta f)g & = & x^{2k}H2l(2i + M + 2l - 2)x^{2l-2}K \\ & & -2k(2j + M + 2k - 2)x^{2k-2}Hx^{2l}K \\ & = & \text{constant} \times x^{2k+2l-2}HK. \end{array}$$

It now suffices to remark that

$$\int_{SB} x^{2k+2l-2}HK = 0,$$

again using the orthogonality of spherical harmonics of different degree.

2) The case where $i = j$. Then the left-hand side becomes:

$$\begin{aligned}
\int_{SS} (f\mathbb{E}g - (\mathbb{E}f)g) &= (2l-2k)\int_{SS} x^{2k+2l}HK \\
&= -(2l-2k)\int_{SS} x^{2k+2l-2}HK \\
&= -(2l-2k)\frac{(-1)^{k+l+i-1}2\pi^{M/2}}{2^{2(k+l+i-1)}(k+l+i-1)!} \\
&\quad \times \frac{\Delta^{k+l+i-1}x^{2k+2l-2}HK}{\Gamma(k+l+i-1+M/2)}
\end{aligned}$$

and the right-hand side:

$$\begin{aligned}
\int_{SB} (f\Delta g - (\Delta f)g) &= (2l-2k)(2i+M+2l+2k-2) \\
&\quad \times \int_{SB} x^{2k+2l-2}HK \\
&= (2l-2k)(2i+M+2l+2k-2)(-1)^{k+l+i-1} \\
&\quad \times \frac{\pi^{M/2}}{2^{2(k+l+i-1)}(k+l+i-1)!} \\
&\quad \times \frac{\Delta^{k+l+i-1}x^{2k+2l-2}HK}{\Gamma(k+l+i-1+M/2+1)}.
\end{aligned}$$

Both sides are obviously equal. □

5.2.6 Funk-Hecke theorem in superspace

Let us first introduce some numerical coefficients, which will be needed later on. Assume for a moment that $M \geq 2$ and put

$$\alpha_l(t^k) = \sigma_{M-1}\int_{-1}^{1} t^k P_l^M(t)(1-t^2)^{\theta}dt$$

with

$$P_n^M(t) = \frac{(-1)^n}{2^n(\theta+1)(\theta+2)\dots(\theta+n)}(1-t^2)^{-\theta}\frac{d^n}{dt^n}(1-t^2)^{\theta+n}$$

the Legendre polynomial of degree n in M dimensions and $\theta = (M-3)/2$. Using partial integration and the definition of the Gamma function we obtain the following explicit expression for $\alpha_l(t^k)$:

$$\begin{aligned}
\alpha_l(t^k) &= \frac{k!}{(k-l)!} \frac{2\pi^{\frac{M-1}{2}}}{2^l} \frac{\Gamma(\frac{k-l+1}{2})}{\Gamma(\frac{M+k+l}{2})} \quad \text{if } k+l \text{ even and } k \geq l \\
&= 0 \quad \text{if } k+l \text{ odd} \\
&= 0 \quad \text{if } k < l.
\end{aligned}$$

Note that the resulting formulae still make sense when $M < 2$. If $M = -2u$ ($u = 0, 1, 2, \ldots$), we have to replace the formula for $k+l$ even by

$$\begin{aligned}
\alpha_l(t^k) &= \frac{k!}{(k-l)!} \frac{2\pi^{\frac{M-1}{2}}}{2^l} \frac{\Gamma(\frac{k-l+1}{2})}{\Gamma(\frac{M+k+l}{2})} \quad \text{if } k+l > 2u. \\
&= 0 \quad \text{if } k+l \leq 2u.
\end{aligned}$$

Finally we also need the following coefficients, where $M = -2u$ and $k+l \leq 2u$:

$$\begin{aligned}
\alpha_l^*(t^k) &= \frac{k!(u-\frac{k+l}{2})!}{(k-l)!u!} \frac{\pi^{-\frac{1}{2}}(-1)^{\frac{k+l}{2}}}{2^l} \Gamma(\frac{k-l+1}{2}) \quad \text{if } k+l \text{ even, } k \geq l \\
&= 0 \quad \text{if } k+l \text{ odd} \\
&= 0 \quad \text{if } k < l.
\end{aligned}$$

We have the following technical lemma.

Lemma 15. *The coefficients $\alpha_l(t^k)$ and $\alpha_l^*(t^k)$ satisfy the following recursion relation:*

$$\begin{aligned}
\alpha_l(t^k) &= \frac{k}{4s(s+M/2-1)} \left((k-1)\alpha_l(t^{k-2}) + 2l\alpha_{l-1}(t^{k-1})\right) \\
\alpha_l^*(t^k) &= \frac{k}{4s(s+M/2-1)} \left((k-1)\alpha_l^*(t^{k-2}) + 2l\alpha_{l-1}^*(t^{k-1})\right)
\end{aligned}$$

where $2s = k+l$.

Proof. This follows from a careful comparison between both sides of the formulae, using the properties of the Gamma function. $\square$

Furthermore we note that α_l can be extended by linearity to a functional on the space of polynomial functions $f(t)$ in one variable t.

Finally we put

$$\langle x, y\rangle = \frac{1}{2}(xy+yx) = -\sum_{i=1}^{m} x_i y_i + \frac{1}{2}\sum_{j=1}^{n}(\grave{x}_{2j-1}\grave{y}_{2j} - \grave{x}_{2j}\grave{y}_{2j-1})$$

where x and y are two independent vector variables.

We can now prove the following theorem (see e.g. [64] for a classical version).

Theorem 32 (Funk-Hecke)**.** *Let $f(t)$ be a polynomial in one variable. Let x, y be independent vector variables satisfying $x^2 = y^2 = -1$. Let $H_l \in \mathcal{H}_l$ be a spherical harmonic of degree l. Then*

$$\int_{SS} f(-\langle x,y\rangle) H_l(x) = \alpha_l(f) H_l(y)$$

with $\alpha_l(f)$ defined as above.

Proof. We first examine the case where $M \not\in -2\mathbb{N}$. It suffices to prove the theorem for $f = t^k$, $k \in \mathbb{N}$. We calculate that

$$\begin{aligned}\Delta\left(\langle x,y\rangle^k H_l\right) &= -k(k-1)\langle x,y\rangle^{k-2}H_l\\ &\quad +2k\langle x,y\rangle^{k-1}\left(\sum_i y_i\partial_{x_i} + \sum_j \grave{y}_j\partial_{\grave{x}_j}\right)H_l,\end{aligned}$$

taking into account that $y^2 = -1$. We now use induction on k. We first examine the case $k = 1$, the case $k = 0$ being trivial. Then using the previous relation our integral becomes (we assume $l + 1 = 2t$, the odd case being trivial)

$$\begin{aligned}&\int_{SS}(-\langle x,y\rangle)H_l(x)\\ =\ & -\frac{2\pi^{M/2}}{2^{2t}t!\Gamma(t+M/2)}(-1)^t\Delta^t\left(\langle x,y\rangle H_l\right)\\ =\ & \frac{2\pi^{M/2}}{2^{2t}t!\Gamma(t+M/2)}(-1)^{t-1}\Delta^{t-1}2\left(\sum_i y_i\partial_{x_i} + \sum_j \grave{y}_j\partial_{\grave{x}_j}\right)H_l\\ =\ & \delta_{1l}\frac{2\pi^{M/2}}{2^2\Gamma(1+M/2)}2\left(\sum_i y_i\partial_{x_i} + \sum_j \grave{y}_j\partial_{\grave{x}_j}\right)H_l.\end{aligned}$$

Now $H_l = H_1$ has the following general form

$$H_1(x) = \sum_i a_i x_i + \sum_j b_j \grave{x}_j, \quad a_i, b_j \in \mathbb{R}$$

because $\mathcal{H}_1 = \mathcal{P}_1$, so

$$\int_{SS} (-\langle x, y\rangle) H_l(x) = \delta_{1l} \frac{\pi^{M/2}}{\Gamma(1+M/2)} H_l(y)$$

which is equal to the right-hand side of the formula to be proven.

Let us now consider the induction step. Suppose the theorem holds for all $i < k$, i.e.

$$\int_{SS} (-\langle x, y\rangle)^i H_l(x) = \alpha_l(t^i) H_l(y), \quad \text{for all } l$$

then we prove the theorem for t^k. We assume that $k + l = 2s$, otherwise both sides are zero. Now we have

$$\begin{aligned}
& \int_{SS} (-\langle x, y\rangle)^k H_l(x) \\
= \quad & \frac{2\pi^{M/2}}{2^{2s} s! \Gamma(s + M/2)} (-1)^{k+s} \Delta^s \left(\langle x, y\rangle^k H_l\right) \\
= \quad & \frac{2\pi^{M/2}}{2^{2s} s! \Gamma(s + M/2)} (-1)^{s-1} \Delta^{s-1} k(k-1)(-1)^{k-2} \langle x, y\rangle^{k-2} H_l \\
+ \quad & \frac{2\pi^{M/2}}{2^{2s} s! \Gamma(s + M/2)} (-1)^{s-1} 2k (-1)^{k-1} \\
& \times \Delta^{s-1} \langle x, y\rangle^{k-1} \left(\sum_i y_i \partial_{x_i} + \sum_j \grave{y}_j \partial_{\grave{x}_j} \right) H_l \\
= \quad & \frac{k(k-1)}{4s(s + M/2 - 1)} \int_{SS} (-\langle x, y\rangle)^{k-2} H_l(x) \\
+ \quad & \frac{2k}{4s(s + M/2 - 1)} \int_{SS} (-\langle x, y\rangle)^{k-1} \left(\sum_i y_i \partial_{x_i} + \sum_j \grave{y}_j \partial_{\grave{x}_j} \right) H_l \\
= \quad & H_l(y) \frac{k}{4s(s + M/2 - 1)} \left((k-1)\alpha_l(t^{k-2}) + 2l\alpha_{l-1}(t^{k-1}) \right) \\
= \quad & \alpha_l(t^k) H_l(y)
\end{aligned}$$

where we have used lemma 15 and the induction hypothesis.

We now discuss the case where $M = -2u$. This is slightly more difficult, because now the first u terms in $\int_{SS}$ vanish, making the induction procedure more complicated.

First one has to prove the theorem for the purely fermionic integral (see formula (5.8)), using the coefficients $\alpha_l^*(t^k)$ (which is completely similar to the above). This yields the following formula

$$\Delta^s(-\langle x, y\rangle)^k H_l(x) = \frac{2^{2s} s! u!}{(u-s)!} \alpha_l^*(t^k) H_l(y)$$

with $k + l = 2s$ and $s \leq u$. This formula is also valid if $M = -2u$ with $m \neq 0$. We then use this formula as a first step in the induction proof for $\int_{SS}$, because the following identity holds

$$\alpha_l(t^k) = \frac{(-1)^u k u! \pi^{M/2}}{2(u+1)} \left((k-1)\alpha_l^*(t^{k-2}) + 2l\alpha_{l-1}^*(t^{k-1})\right)$$

with $k + l = 2u + 2$. $\square$

Remark 18. *Note that our theorem is of course also valid in the case $n = 0$, although we used a completely different method as in other textbooks. It would also be possible to use our method to prove the Funk-Hecke theorem for Dunkl harmonics established in [104].*

As a consequence, we immediately have the following

Corollary 4. *Let $f(t)$ be a polynomial in one variable. Let x, y satisfy $x^2 = y^2 = -1$. Then*

$$\int_{SS} f(-\langle x, y\rangle) = \alpha_0(f(t)).$$

Remark 19. *Classically this result is easily obtained by realizing that a function depending on the inner product of x and y is constant on each hyperplane perpendicular to y.*

Using the previous theorem, we are now able to determine the reproducing kernel for spaces of spherical harmonics.

Corollary 5 (Reproducing kernel)**.** *Let $M > 1$. Then*

$$F_k(x, y) = \frac{N(M, k)}{\sigma_M} P_k^M(-\langle x, y\rangle)$$

is a reproducing kernel for the space $\mathcal{H}_k$, i.e.

$$\int_{SS} F_k(x,y)H_l(x) = \delta_{kl}H_l(y), \quad \textit{for all } H_l \in \mathcal{H}_l,$$

where

$$N(M,k) = \frac{2k+M-2}{k}\left(\begin{array}{c} k+M-3 \\ k-1 \end{array}\right).$$

Proof. Using the Funk-Hecke theorem we obtain

$$\begin{aligned} & \int_{SS} P_k^M(-\langle x,y\rangle)H_l(x) \\ = \quad & H_l(y)\sigma_{M-1}\int_{-1}^{1} P_k^M(t)P_l^M(t)(1-t^2)^{\frac{M-3}{2}}dt. \end{aligned}$$

Now the orthogonality-relation of the Legendre-polynomials (see [64]) yields

$$\int_{-1}^{1} P_k^M(t)P_l^M(t)(1-t^2)^{\frac{M-3}{2}}dt = \delta_{kl}\frac{\sigma_M}{\sigma_{M-1}N(M,k)}$$

so

$$\int_{SS} P_k^M(-\langle x,y\rangle)H_l(x) = \delta_{kl}H_l(y)\frac{\sigma_M}{N(M,k)}$$

which completes the proof. □

Remark 20. *It is also possible to construct a reproducing kernel for spaces of spherical monogenics $\mathcal{M}_k$, using the same strategy as in [38].*

The real significance of the Funk-Hecke theorem is that spaces of spherical harmonics are eigenspaces of zonal integral transformations, i.e. transformations whose kernel depends on the inner product of two generalized points on the supersphere. We consider a simple application of this idea.

Application: the super spherical Fourier transform.

Consider the kernel e^{iat}, where for now $a \in \mathbb{R}$. Let us first calculate the coefficients $\alpha_l(e^{iat})$. We have

$$e^{iat} = \sum_{k=0}^{\infty} \frac{(iat)^k}{k!}$$

We only need to consider the case $k \geq l$ and $k+l$ even, since the other terms are zero. So, putting $k = l + 2s$, we calculate:

$$\begin{aligned}
\alpha_l(e^{iat}) &= \sum_{s=0}^{\infty} \alpha_l(\frac{(iat)^{l+2s}}{(l+2s)!})\\
&= \sum_{s=0}^{\infty} \frac{i^{l+2s}}{(l+2s)!} \frac{(l+2s)!}{(2s)!} \frac{2\pi^{\frac{M-1}{2}}}{2^l} \frac{\Gamma(s+1/2)}{\Gamma(M/2+l+s)} a^{l+2s}\\
&= \sum_{s=0}^{\infty} \frac{i^l}{2^l} 2\pi^{\frac{M}{2}} \frac{(-1)^s}{s! 2^{2s} \Gamma(M/2+l+s)} a^{l+2s}\\
&= i^l (2\pi)^{M/2} a^{1-M/2} J_{\frac{M}{2}+l-1}(a),
\end{aligned}$$

with $J_\alpha(x)$ the Bessel function of the first kind of order α.

Now let us introduce the following operator ($a = 1$):

$$\mathcal{F}_{SS}(.)(y) = \int_{SS} \exp(-i\langle x, y\rangle)(.)$$

which we will call the super spherical Fourier transform. Then we obtain the following

Theorem 33. *Let $H_l \in \mathcal{H}_l$. Then*

$$\mathcal{F}_{SS}(H_l(x))(y) = i^l (2\pi)^{M/2} J_{\frac{M}{2}+l-1}(1) H_l(y).$$

In chapter 7 we will make a detailed study of the general Fourier transform in superspace.

5.3 Integration over superspace and connection with the Berezin integral

We can combine the results of the previous section with the concept of integration in spherical co-ordinates in Euclidean space in order to obtain a definition of an integral over the whole superspace. Indeed, using spherical co-ordinates the integral of a function f over $\mathbb{R}^m$ can be expressed as follows:

$$\int_{\mathbb{R}^m} f(\underline{x}) dV(\underline{x}) = \int_0^{+\infty} r^{m-1} dr \int_{\mathbb{S}^{m-1}} f(r\underline{\xi}) d\sigma(\underline{\xi}),$$

where $\underline{x} \in \mathbb{R}^m$ is written as $\underline{x} = r\underline{\xi}$ with $\underline{\xi} \in \mathbb{S}^{m-1}$.

It is possible to extend this recipe to superspace by substituting M for m. This is justified by the calculations in chapter 4, where we introduced a formal inner product on the space of polynomials using the same substitution. If we now consider a function of the following form:

$$f = R_k \exp(x^2), \quad R_k \in \mathcal{P}_k$$

we then obtain the following *definition* for an integral $\int_{\mathbb{R}^{m|2n}}$ over the whole superspace:

$$\begin{aligned}\int_{\mathbb{R}^{m|2n}} f &= \int_0^{+\infty} r^{k+M-1} e^{-r^2} dr \int_{SS} R_k \\ &= \frac{1}{2}\Gamma(\frac{k+M}{2}) \int_{SS} R_k\end{aligned}$$

where the second expression is used if the integral over r is divergent. Using the definition of the integral over the supersphere (see formula (5.7)) we immediately arrive at the following

Theorem 34. *The integral of a function $f = R\exp(x^2)$, with $R \in \mathcal{P}$ an arbitrary superpolynomial, is given by the following formula:*

$$\int_{\mathbb{R}^{m|2n}} f = \sum_{k=0}^{\infty} (-1)^k \frac{\pi^{M/2}}{2^{2k} k!} (\Delta^k R)(0) = \pi^{M/2} (\exp(-\Delta/4)) R(0). \tag{5.9}$$

As any reference to the super-dimension M has disappeared in formula (5.9) (except in the scaling of the formula), it seems interesting to compare this definition with the Berezin integral. This integral is defined as follows (see e.g. [8, 73]). Let f be an element of $C^\infty(\mathbb{R}^m) \otimes \Lambda_{2n}$, i.e. f is a superfunction with the following expansion:

$$f(x, \grave{x}) = \sum_{\nu=(\nu_1,\ldots,\nu_{2n})} f_\nu(x) \grave{x}_1^{\nu_1} \ldots \grave{x}_{2n}^{\nu_{2n}}$$

where $\nu_i = 0$ or 1 and $f_\nu(x)$ is a smooth function of the (real) co-ordinates $(x_1, \ldots, x_m)$. Then, by definition, the Berezin integral is given by

$$\int_{\mathbb{R}^m} dV(\underline{x}) \int_B f$$

where

$$\int_B = \pi^{-n}\partial_{\grave{x}_{2n}}\ldots\partial_{\grave{x}_1} = \frac{(-1)^n\pi^{-n}}{4^n n!}\partial_{\underline{\grave{x}}}^{2n}.$$

This means that the Berezin integral of f is given by

$$\pi^{-n}\int_{\mathbb{R}^m} dV(\underline{x}) f_{(1,\ldots,1)}.$$

Now we have the following theorem.

Theorem 35. *For functions f of the form $R\exp(x^2)$, with R a polynomial, the Berezin integral is equivalent with the integral defined in (5.9), i.e.*

$$\int_{\mathbb{R}^{m|2n}} f = \int_{\mathbb{R}^m} dV(\underline{x}) \int_B f. \tag{5.10}$$

Proof. First note that it suffices to give the proof for functions f of the following form

$$\begin{aligned} f &= R_{2k}\exp x^2 \\ &= x_1^{2\alpha_1}\ldots x_m^{2\alpha_m}(\grave{x}_1\grave{x}_2)^{\beta_1}\ldots(\grave{x}_{2n-1}\grave{x}_{2n})^{\beta_n}\exp(x^2), \end{aligned}$$

where $\alpha_i \in \mathbb{N}$, $\beta_i \in \{0,1\}$, $\sum\alpha_i + \sum\beta_i = k$ and $\sum\beta_i = l$. We will now calculate the integral of f with the two definitions and show that we obtain the same result. Let us first calculate $\int_{\mathbb{R}^{m|2n}} f$; we need to calculate $\Delta^k(R_{2k})$. One immediately sees that in Δ^k only the term

$$(-1)^{k-l}2^{2l}\partial_{x_1}^{2\alpha_1}\ldots\partial_{x_m}^{2\alpha_m}(\partial_{\grave{x}_1}\partial_{\grave{x}_2})^{\beta_1}\ldots(\partial_{\grave{x}_{2n-1}}\partial_{\grave{x}_{2n}})^{\beta_n}$$

is non-vanishing. As this term occurs $\frac{k!}{\alpha_1!\ldots\alpha_m!}$ times, we obtain

$$\Delta^k(R_{2k}) = \frac{k!}{\alpha_1!\ldots\alpha_m!}2^{2l}(2\alpha_1)!\ldots(2\alpha_m)!(-1)^k.$$

Using this result we find that

$$\begin{aligned} \int_{\mathbb{R}^{m|2n}} f &= (-1)^k\frac{\pi^{M/2}}{2^{2k}k!}(\Delta^k R_{2k})(0) \\ &= \frac{\pi^{M/2}}{2^{2k}k!}\frac{k!}{\alpha_1!\ldots\alpha_m!}2^{2l}(2\alpha_1)!\ldots(2\alpha_m)! \\ &= \pi^{-n}\frac{\pi^{m/2}}{2^{2k-2l}}\frac{(2\alpha_1)!\ldots(2\alpha_m)!}{\alpha_1!\ldots\alpha_m!} \\ &= \pi^{-n}\int_{\mathbb{R}^m} x_1^{2\alpha_1}\ldots x_m^{2\alpha_m}\exp(\underline{x}^2)\,dV(\underline{x}). \end{aligned}$$

On the other hand let us consider the Berezin integral of f. We need to determine the term of f in $\grave{x}_1 \ldots \grave{x}_{2n}$. This is equivalent with determining the term of $\exp(x^2)$ in $(\grave{x}_1\grave{x}_2)^{1-\beta_1} \ldots (\grave{x}_{2n-1}\grave{x}_{2n})^{1-\beta_n}$. Now we calculate

$$\begin{aligned}
\exp(x^2) &= \exp(\underline{x}^2 + \underline{\grave{x}}^2) \\
&= \exp(\underline{x}^2)\exp(\underline{\grave{x}}^2) \\
&= \exp(\underline{x}^2)(\sum_{k=0}^{n} \frac{\underline{\grave{x}}^{2k}}{k!}) \\
&= \exp(\underline{x}^2)((\grave{x}_1\grave{x}_2)^{1-\beta_1} \ldots (\grave{x}_{2n-1}\grave{x}_{2n})^{1-\beta_n} + \text{other terms}).
\end{aligned}$$

So

$$R_{2k}\exp(x^2) = x_1^{2\alpha_1} \ldots x_m^{2\alpha_m} \exp(\underline{x}^2)\grave{x}_1 \ldots \grave{x}_{2n} + l.o.t.$$

By the definition of the Berezin integral we now find

$$\pi^{-n} \int_{\mathbb{R}^m} x_1^{2\alpha_1} \ldots x_m^{2\alpha_m} \exp(\underline{x}^2)\, dV(\underline{x})$$

which is the same result as calculated with $\int_{\mathbb{R}^{m|2n}}$. □

Remark 21. *Note that the set of functions $f = R(x)\exp(x^2)$, considered in theorem 35, is dense in e.g. $\mathcal{S} \otimes \Lambda_{2n}$, with $\mathcal{S}$ the space of rapidly decreasing functions in $\mathbb{R}^m$. It is then easy to check that the Berezin integral*

$$\int_{\mathbb{R}^m} dV(\underline{x}) \int_B$$

considered in theorem 35 is the unique continuous extension of the integral

$$\int_{\mathbb{R}^{m|2n}}$$

which we defined for all functions of the type $f = R(x)\exp(x^2)$, $R(x) \in \mathcal{P}$. Similar conclusions hold for various other types of function spaces.

In this section we have thus obtained a new way of defining integration on superspace for a sufficiently large set of functions, without resorting to the ad hoc formulation of Berezin.

Remark 22. *We can of course also combine the idea of integration in spherical co-ordinates with one of the other possibilities of integration over the supersphere. In the case we would e.g. use formula (5.5), we would find the following formula for integration over the whole superspace*

$$\int_{1,\mathbb{R}^{m|2n}} R \exp(x^2) = \frac{c}{2} \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{4^{k+1}(k-1)!(k+M/2)} \left(\Delta_b (2M\Delta^{k-1} - x^2 \Delta^k) R \right)(0).$$

Note that in this case the super-dimension M still appears in the resulting formula.

Chapter 6

Fundamental solutions

In the previous chapters we have worked mainly with polynomial functions in the commuting and anti-commuting variables. In this chapter we will consider more general superfunctions and some of their properties. In the first section we will give some definitions needed in the sequel and study some basic function-theoretical properties. In the next sections we will determine the fundamental solutions for all natural powers of the super Dirac operator. Note that we will reobtain these fundamental solutions in the next chapter using Fourier methods. However, the Fourier method can only be used to study operators that have 'good' Fourier symbols, whereas the present method is valid for a larger class of differential operators.

The results of this chapter have been published in [33].

6.1 Some function-theoretical results

We will denote the space of harmonic functions, i.e. null-solutions of the super Laplace operator, by $\mathcal{H}(\Omega)_{m|2n} \subset C^2(\Omega) \otimes \Lambda_{2n}$, where Ω is an open set in $\mathbb{R}^m$. Similarly we denote by $\mathcal{M}(\Omega)^{l(r)}_{m|2n} \subset C^1(\Omega)_{m|2n}$ the space of left (resp. right) monogenic functions, i.e. left (resp. right) null-solutions of the super Dirac operator. We clearly have that $\mathcal{M}(\Omega)^{l(r)}_{m|2n} \subset \mathcal{H}(\Omega)_{m|2n} \otimes \mathcal{C}$.

The following theorem generalizes a classical result of harmonic analysis.

Theorem 36. *Null-solutions of the super Laplace and the super Dirac operator* *are C^∞-functions, i.e.*

$$\begin{array}{rcl}\mathcal{H}(\Omega)_{m|2n} \otimes \mathcal{C} & \subset & C^\infty(\Omega)_{m|2n} \\ \mathcal{M}(\Omega)_{m|2n}^{l(r)} & \subset & C^\infty(\Omega)_{m|2n}.\end{array}$$

Proof. It suffices to give the proof for harmonic functions, as monogenic functions are also harmonic. So we consider a function $g \in \mathcal{H}(\Omega)_{m|2n}$. Such a function can be written as

$$g = \sum_{(\alpha)} g_{(\alpha)} \grave{x}_1^{\alpha_1} \dots \grave{x}_{2n}^{\alpha_{2n}}$$

with $(\alpha) = (\alpha_1, \dots, \alpha_{2n})$, $\alpha_i \in \{0, 1\}$, $|(\alpha)| = \sum_i \alpha_i$ and $g_{(\alpha)} \in C^2(\Omega)$. Expressing that $\Delta g = \Delta_b g + \Delta_f g = 0$ leads to a set of equations of the following type

$$\Delta_b g_{(\alpha)} = \sum_{(\beta)} c_{(\beta)} g_{(\beta)}, \quad |(\beta)| = |(\alpha)| + 2$$

with $c_{(\beta)} \in \mathbb{R}$. We conclude that for every (α) there exists a $k \in \mathbb{N}$ $(k \leq n+1)$ such that $\Delta_b^k g_{(\alpha)} = 0$. Hence $g_{(\alpha)}$ is polyharmonic and thus an element of $C^\infty(\Omega)$. $\square$

We can define a super Dirac distribution $\delta(x) \in \mathcal{D}'(\mathbb{R}^m) \otimes \Lambda_{2n}$ by

$$\delta(x) = \pi^n \delta(\underline{x}) \grave{x}_1 \dots \grave{x}_{2n} = \pi^n \delta(\underline{x}) \frac{\underline{\grave{x}}^{2n}}{n!}$$

with $\delta(\underline{x})$ the classical Dirac distribution in $\mathbb{R}^m$. We clearly have that

$$\begin{array}{rcl}\langle \delta(x-y), f(x) \rangle & = & \pi^n \displaystyle\int_{\mathbb{R}^{m|2n}} \delta(\underline{x} - \underline{y})(\grave{x}_1 - \grave{y}_1) \dots (\grave{x}_{2n} - \grave{y}_{2n}) f(x) \\ & = & f(y)\end{array}$$

with $f \in \mathcal{D}(\mathbb{R}^m)_{m|2n}$. Note that the Dirac distribution is even (due to the fact that we consider $2n$ anti-commuting variables) and hence commutes with all functions.

Finally, we define the convolution of two functions f and g by

$$f * g(u) = \int_{\mathbb{R}^{m|2n}, x} f(u - x) g(x),$$

where the subindex x means that we are integrating with respect to the x-variables. This operation is clearly not commutative; however note that

$$\begin{aligned} f * g(u) &= \int_{\mathbb{R}^{m|2n},x} f(u-x)g(x) \\ &= \int_{\mathbb{R}^{m|2n},x} \int_{\mathbb{R}^{m|2n},y} \delta(y+x-u)f(y)g(x) \\ &= \int_{\mathbb{R}^{m|2n},y} f(y) \int_{\mathbb{R}^{m|2n},x} \delta(y+x-u)g(x) \\ &= \int_{\mathbb{R}^{m|2n},y} f(y)g(u-y). \end{aligned}$$

6.2 Fundamental solutions in $\mathbb{R}^m$

The fundamental solutions for the natural powers of the classical Laplace operator Δ_b in $\mathbb{R}^m$ are very well known, see e.g. [1]. We denote by $\nu_{2l}^{m|0}$, $l = 1, 2, \ldots$ a sequence of such fundamental solutions, satisfying

$$\begin{aligned} \Delta_b^j \nu_{2l}^{m|0} &= \nu_{2l-2j}^{m|0}, \quad j < l \\ \Delta_b^l \nu_{2l}^{m|0} &= \delta(\underline{x}). \end{aligned}$$

Their explicit form depends both on the dimension m and on l. More specifically, in the case where m is odd we have

$$\nu_{2l}^{m|0} = \frac{r^{2l-m}}{\gamma_{l-1}}, \qquad \gamma_l = (-1)^{l+1}(2-m)4^l l! \frac{\Gamma(l+2-m/2)}{\Gamma(2-m/2)} \frac{2\pi^{m/2}}{\Gamma(m/2)} \tag{6.1}$$

with $r = \sqrt{-\underline{x}^2}$. The formulae for m even are more complicated and can e.g. be found in [1].

Concerning the refinement to Clifford analysis, we clearly have that $\nu_{2l+1}^{m|0} = -\partial_{\underline{x}} \nu_{2l+2}^{m|0}$ is a fundamental solution of $-\Delta_b^l \partial_{\underline{x}} = (-\partial_{\underline{x}})^{2l+1}$. (The minus sign is necessary because the restriction of the super Dirac operator ∂_x to the purely bosonic case is given by $-\partial_{\underline{x}}$.)

6.3 Fundamental solution of Δ and ∂_x

From now on we restrict ourselves to the case where $m \neq 0$. The purely fermionic case will be discussed briefly in section 6.6.

Our aim is to construct a function ρ such that in distributional sense

$$\Delta\rho = \delta(x).$$

We propose the following form for the fundamental solution:

$$\rho = \sum_{k=0}^{n} a_k (\Delta_b^{n-k}\phi)\underline{x}^{2n-2k},$$

with ϕ and $a_k \in \mathbb{R}$ to be determined.
Now let us calculate $\Delta\rho$

$$\begin{aligned} \Delta\rho &= (\Delta_b + \Delta_f)\rho \\ &= \sum_{k=0}^{n} a_k (\Delta_b^{n-k+1}\phi)\underline{x}^{2n-2k} \\ &\quad + \sum_{k=1}^{n} a_k (2n-2k)(2n-2k-2-2n)(\Delta_b^{n-k}\phi)\underline{x}^{2n-2k-2} \\ &= a_0(\Delta_b^{n+1}\phi)\underline{x}^{2n} + \sum_{k=1}^{n} \left[a_k - 2k(2n-2k+2)a_{k-1}\right](\Delta_b^{n-k+1}\phi)\underline{x}^{2n-2k}. \end{aligned}$$

So ρ is a fundamental solution if and only if

$$a_0(\Delta_b^{n+1}\phi) = \delta(\underline{x})\frac{\pi^n}{n!}$$

and a_k satisfies the recurrence relation

$$a_k = 4k(n-k+1)a_{k-1}.$$

The first equation leads to

$$\phi = \nu_{2n+2}^{m|0}, \quad a_0 = \frac{\pi^n}{n!}.$$

We then immediately find the following expression for the a_k:

$$a_k = \pi^n \frac{4^k k!}{(n-k)!}, \qquad k = 0, \ldots, n.$$

Summarizing we obtain the following theorem.

Theorem 37. *The function* $\nu_2^{m|2n}$ *defined by*

$$\nu_2^{m|2n} = \pi^n \sum_{k=0}^{n} \frac{4^k k!}{(n-k)!} \nu_{2k+2}^{m|0} \underline{x}^{2n-2k},$$

with $\nu_{2k+2}^{m|0}$ *as in section 6.2, is a fundamental solution for the operator* Δ.

Proof. It is clear that $\nu_2^{m|2n} \in L_1^{\mathrm{loc}}(\mathbb{R}^m)_{m|2n}$. Moreover, we also have that $\nu_2^{m|2n} \in \mathcal{H}(\mathbb{R}^m - \{0\})_{m|2n}$ and that $\Delta \nu_2^{m|2n} = \delta(x)$ in distributional sense. $\square$

Now suppose that m is odd, then the fundamental solution is given explicitly by

$$\nu_2^{m|2n} = \pi^n \frac{\Gamma(m/2)}{2(2-m)\pi^{m/2}} \sum_{k=0}^{n} \frac{(-1)^{k+1}}{(n-k)!} \frac{\Gamma(2-m/2)}{\Gamma(k+2-m/2)} r^{2k+2-m} \underline{x}^{2n-2k}, \quad (6.2)$$

where we have used formula (6.1).
As $\Delta \nu_2^{m|2n} = \delta(x)$, a fundamental solution for the Dirac operator ∂_x is given by $\partial_x \nu_2^{m|2n}$. This leads to the following

Theorem 38. *The function* $\nu_1^{m|2n}$ *defined by*

$$\nu_1^{m|2n} = \pi^n \sum_{k=0}^{n-1} 2 \frac{4^k k!}{(n-k-1)!} \nu_{2k+2}^{m|0} \underline{x}^{2n-2k-1} + \pi^n \sum_{k=0}^{n} \frac{4^k k!}{(n-k)!} \nu_{2k+1}^{m|0} \underline{x}^{2n-2k}$$

is a fundamental solution for the operator ∂_x.

Remark 23. *We could have proposed the following form for the fundamental solution of* Δ*:*

$$g = \frac{1}{(x^2)^{\frac{M-2}{2}}}, \quad (6.3)$$

where we have substituted the super-dimension M *for* m *in the classical expression. This technique is inspired by radial algebra (see [96]), which gives a very general framework for constructing theories of Clifford analysis, based on the introduction of an abstract dimension parameter (in this case the super-dimension). This leads partially to the correct result (see also [99]). Indeed,* *formally we can expand formula (6.3) as follows:*

$$\begin{aligned} g &= \frac{1}{(x^2)^{\frac{M-2}{2}}} \\ &= \frac{1}{(\underline{x}^2 + \underline{\grave{x}}^2)^{\frac{M-2}{2}}} \\ &= \frac{1}{(\underline{x}^2)^{\frac{M-2}{2}}} \left(1 + \frac{\underline{\grave{x}}^2}{\underline{x}^2}\right)^{1-\frac{M}{2}} \\ &= \sum_{k=0}^{n} \binom{1-\frac{M}{2}}{k} \frac{\underline{\grave{x}}^{2k}}{(\underline{x}^2)^{\frac{M}{2}-1+k}}. \end{aligned}$$

The coefficients in this development are proportional to the ones obtained in formula (6.2), so this yields the correct result. The expansion is however only valid if m is odd, since for m even the fundamental solutions $\nu_{2k+2}^{m|0}$ take a more complicated form involving logarithmic terms.

The fundamental solution can of course be used to determine solutions of the inhomogeneous equation $\Delta f = \rho$. We have for example the following proposition.

Proposition 4. *Let $\rho \in \mathcal{D}(\Omega)_{m|2n}$, then a solution of $\Delta f = \rho$ is given by*

$$f(x) = \nu_2^{m|2n} * \rho = \int_{\mathbb{R}^{m|2n}} \nu_2^{m|2n}(x-y)\rho(y).$$

6.4 Fundamental solution of Δ^k and $\Delta^k \partial_x$, $k \in \mathbb{N}$

A similar technique as in section 6.3 can be used for the polyharmonic case. First we expand Δ^k as

$$\Delta^k = \sum_{j=0}^{k} \binom{k}{j} \Delta_b^{k-j} \Delta_f^j.$$

This expansion is valid since Δ_b commutes with Δ_f.
Now we propose the following form for the fundamental solution of Δ^k:

$$\rho = \sum_{l=0}^{n} a_l \left(\Delta_b^{n-l} \phi\right) \underline{\grave{x}}^{2n-2l}$$

with ϕ and $a_l \in \mathbb{R}$ still to be determined. We calculate that

$$\Delta^k \rho = \sum_{l=0}^{n} a_l \sum_{j=0}^{k} \binom{k}{j} \left(\Delta_b^{n-l+k-j}\phi\right) \Delta_f^j \underline{x}^{2n-2l}.$$

Using lemma 3 we have

$$\Delta_f^j \underline{x}^{2n-2l} = 4^j(-1)^j \frac{(n-l)!}{(n-l-j)!}\frac{(l+j)!}{l!}\underline{x}^{2n-2l-2j}, \quad j \leq n-l$$

yielding

$$\Delta^k \rho = \sum_{l=0}^{n} a_l \sum_{j=0}^{k} \binom{k}{j} 4^j(-1)^j \frac{(n-l)!}{(n-l-j)!}\frac{(l+j)!}{l!} \left(\Delta_b^{n-l+k-j}\phi\right) \underline{x}^{2n-2l-2j}.$$

Putting $\Delta^k\rho = \delta(x)$ leads to the following set of equations

$$a_0 \Delta_b^{n+k}\phi = \delta(\underline{x})\frac{\pi^n}{n!} \tag{6.4}$$

$$\sum_{j=0}^{k} a_{l-j}\binom{k}{j} 4^j(-1)^j \frac{(n-l+j)!}{(n-l)!}\frac{l!}{(l-j)!} = 0 \tag{6.5}$$

$$a_{-1} = a_{-2} = a_{-3} = \ldots = 0. \tag{6.6}$$

We immediately have that $a_0 = \pi^n/n!$ and that $\phi = \nu_{2n+2k}^{m|0}$. Equations (6.5) and (6.6) can be simplified by means of the substitution

$$a_l = \pi^n 4^l \frac{l!}{(n-l)!} b_l$$

which results into

$$\sum_{j=0}^{\min(k,l)} b_{l-j}\binom{k}{j}(-1)^j = 0, \qquad b_0 = 1.$$

The solution to this recurrence equation is (see the subsequent lemma 16)

$$b_l = \binom{l+k-1}{l}.$$

We conclude that

$$a_l = \pi^n 4^l \frac{(l+k-1)!}{(n-l)!(k-1)!}, \quad l = 0, \dots, n.$$

We can summarize the previous results in the following theorem.

Theorem 39. *The function* $\nu_{2k}^{m|2n}$ *defined by*

$$\nu_{2k}^{m|2n} = \pi^n \sum_{l=0}^{n} 4^l \frac{(l+k-1)!}{(n-l)!(k-1)!} \nu_{2l+2k}^{m|0} \underline{x}^{2n-2l},$$

is a fundamental solution for the operator Δ^k.

In a similar vein we obtain the fundamental solution $\nu_{2k+1}^{m|2n}$ for the operator $\Delta^k \partial_x = \partial_x^{2k+1}$ by calculating $\partial_x \nu_{2k+2}^{m|2n}$. This leads to

Theorem 40. *The function* $\nu_{2k+1}^{m|2n}$ *defined by*

$$\begin{aligned} \nu_{2k+1}^{m|2n} &= \pi^n \sum_{l=0}^{n-1} 2 \frac{4^l (l+k)!}{(n-l-1)!k!} \nu_{2l+2k+2}^{m|0} \underline{x}^{2n-2l-1} \\ &\quad + \pi^n \sum_{l=0}^{n} \frac{4^l (l+k)!}{(n-l)!k!} \nu_{2l+2k+1}^{m|0} \underline{x}^{2n-2l}, \end{aligned}$$

is a fundamental solution for the the operator $\Delta^k \partial_x$.

We still have to prove the technical lemma we used in the derivation of theorem 39.

Lemma 16. *The sequence* (b_l), $l = 0, 1, \dots$, *recursively defined by*

$$\sum_{j=0}^{\min(k,l)} b_{l-j} \binom{k}{j} (-1)^j = 0, \quad b_0 = 1$$

is given explicitly by

$$b_l = \binom{l+k-1}{l}.$$

Proof. Define the polynomial $R_l(x)$ by

$$\begin{aligned} R_l(x) &= \sum_{j=0}^{\min(k,l)} (-1)^j \binom{k}{j} \frac{(x+k-j-1)!}{(x-j)!} \\ &= \sum_{j=0}^{\min(k,l)} (-1)^j \binom{k}{j} (x+k-1-j)\dots(x+1-j). \end{aligned}$$

We then have to prove that $R_l(l) = 0$ for all l.
We distinguish between three cases.

1) $l \leq k-2$
We claim that for all $t \leq k-2$

$$R_t(x) = (-1)^t \binom{k-1}{t} (x+k-t-1)\dots(x+1)(x-1)\dots(x-t).$$

This can be proven by using an induction argument. The case where $t=1$ is easily checked. So we suppose the formula holds for $t-1$, then we calculate

$$\begin{aligned} & R_{t-1}(x) + (-1)^t \binom{k}{t} (x+k-1-t)\dots(x+1-t) \\ &= (-1)^t (x+k-t-1)\dots(x+1)(x-1)\dots(x-t+1) \\ &\times \left(\binom{k}{t} x - \binom{k-1}{t-1} (x+k-t) \right) \\ &= (-1)^t \binom{k-1}{t} (x+k-t-1)\dots(x+1)(x-1)\dots(x-t) \\ &= R_t(x) \end{aligned}$$

which proves the hypothesis. Now clearly $R_l(l) = 0$.

2) $l = k-1$
Using the previous results, it is shown that in this case $R_{k-1}(x)$ equals

$$R_{k-1}(x) = -(-1)^k (x-1)\dots(x+1-k)$$

so $R_{k-1}(k-1) = 0$.

3) $l \geq k$
Now we have that

$$R_l(x) = R_k(x) = 0$$

so this case is also proven. □

6.5 A larger class of differential operators

The technique used above can be extended to a larger class of differential operators. Suppose we consider an operator of the following form

$$P = L(x, \partial_x) + \Delta_f$$

with $L(x, \partial_x)$ an elliptic operator in $\mathbb{R}^m$ and Δ_f the fermionic Laplace operator. Note that $L(x, \partial_x)$ and Δ_f commute. An interesting operator in this class is e.g. the super Helmholtz operator $\Delta - \lambda^2$, $\lambda \in \mathbb{R}$ with $L(x, \partial_x) = \Delta_b - \lambda^2$. Denoting by $\mu_{2k}^{m|0}$ $(k = 1, 2, \ldots)$ a set of fundamental solutions for the operators $L(x, \partial_x)^k$ such that

$$\begin{array}{rcl} L(x, \partial_x)^j \mu_{2k}^{m|0} & = & \mu_{2k-2j}^{m|0}, \quad j < k \\ L(x, \partial_x)^k \mu_{2k}^{m|0} & = & \delta(\underline{x}) \end{array}$$

we can now use the same technique as in section 6.4 to obtain a fundamental solution $\mu_{2k}^{m|2n}$ for the operator P^k. This leads to

$$\mu_{2k}^{m|2n} = \pi^n \sum_{l=0}^{n} 4^l \frac{(l + k - 1)!}{(n - l)!(k - 1)!} \mu_{2l+2k}^{m|0} \underline{\grave{x}}^{2n-2l}.$$

6.6 The purely fermionic case

In this case there is no fundamental solution. Indeed, determining the fundamental solution of Δ_f requires solving the algebraic equation

$$\Delta_f \nu_2^{0|2n} = \pi^{-n} \grave{x}_1 \ldots \grave{x}_{2n}$$

which clearly has no solution, since there are no polynomials of degree higher than $2n$ in Λ_{2n}. Equivalently, there is no fundamental solution because Δ is not surjective on Λ_{2n}.

Chapter 7

Integral transforms in superspace

The aim of this chapter is to use the previously developed framework to study generalizations to superspace of the Fourier, the fractional Fourier and the Radon transform. This is of course not a new idea. Several other authors have already developed a Fourier calculus on superspace. Without claiming completeness, we refer the reader to e.g. [84, 47, 66]. Apart from the fact that these authors work in different versions of superanalysis, the main difference is that we use another kernel to define the Fourier transform. Without going into further detail at this point, we will define the fermionic Fourier transform as:

$$\mathcal{F}^{\pm}_{0|2n}(f(x))(y) = (2\pi)^n \int_{B,x} e^{\mp i \langle \underline{\grave{x}}, \underline{\grave{y}} \rangle} f(x)$$

with

$$\langle \underline{\grave{x}}, \underline{\grave{y}} \rangle = \langle \underline{\grave{x}}, \underline{\grave{y}} \rangle_s = \frac{1}{2} \sum_{j=1}^{n} ({\grave{x}}_{2j-1} {\grave{y}}_{2j} - {\grave{x}}_{2j} {\grave{y}}_{2j-1}).$$

The other authors, cited above, use the same formula, but with the following kernel

$$\langle \underline{\grave{x}}, \underline{\grave{y}} \rangle = \langle \underline{\grave{x}}, \underline{\grave{y}} \rangle_o = \sum_{j=1}^{2n} {\grave{x}}_j {\grave{y}}_j .$$

The use of this new kernel is quite crucial for the resulting calculus. First of all, we have that $\langle \underline{\grave{x}}, \underline{\grave{y}} \rangle_s = \langle \underline{\grave{x}}, \underline{\grave{y}} \rangle_s$ whereas $\langle \underline{\grave{x}}, \underline{\grave{y}} \rangle_o = -\langle \underline{\grave{x}}, \underline{\grave{y}} \rangle_o$. Secondly, $\langle \underline{\grave{x}}, \underline{\grave{y}} \rangle_s$

is invariant under symplectic changes of basis, while $\langle \underline{x}\grave{}, \underline{y}\grave{} \rangle_o$ is invariant under the orthogonal group.

The main advantage of our approach is that this Fourier transform behaves nicely with respect to the fermionic Laplace operator (which has the corresponding symplectic structure of the Fourier kernel). This allows for the construction of an eigenbasis of the Fourier transform using the Clifford-Hermite functions, introduced in chapter 4. As a consequence, we obtain an operator exponential expression for the Fourier transform, which enables us to define a fractional Fourier transform (see e.g. [75] and [77] for the classical treatment) in superspace and to study some of its properties.

We are also able to define a Radon transform in superspace by means of the central-slice theorem (see [63, 36]) which connects the classical Radon transform with two consecutive Fourier transformations. Again we will show that this transform behaves nicely with respect to the Clifford-Hermite functions.

The chapter is organized as follows. In section 7.1 we define the fermionic Fourier transform using the symplectic kernel and we study its basic properties. Then in section 7.2 we define the general Fourier transform, we study its eigenfunctions and we establish its operator exponential form. In section 7.3 we apply the general Fourier transform to determine the fundamental solution of the super Laplace operator in a different way. The operator exponential form of the Fourier transform is used in section 7.4 to define the fractional Fourier transform. Finally, the previous results on the Fourier transform are used in section 7.5 to define the Radon transform in superspace.

The results of this chapter have been published in [23].

7.1 The bosonic and fermionic Fourier transform

7.1.1 The bosonic Fourier transform

We define the classical Fourier transform of a function $f \in L_1(\mathbb{R}^m)$ as follows:

$$\mathcal{F}^{\pm}_{m|0}(f)(y) = (2\pi)^{-m/2} \int_{\mathbb{R}^m} dV(\underline{x}) \exp\left(\pm i \sum_{i=1}^{m} x_i y_i\right) f(x).$$

The properties of this transform are very well known, we refer the reader to e.g. [101]. In the sequel we will need the following important theorem.

Theorem 41. *Let $H_l(x) \in \mathcal{H}_l^b$ be a spherical harmonic of degree l. Then one has*

$$\mathcal{F}^{\pm}_{m|0}(H_l(x)\exp(\underline{x}^2/2))(y) = (\pm i)^l H_l(y)\exp(\underline{y}^2/2).$$

Proof. See e.g. [101]. $\square$

7.1.2 The fermionic Fourier transform

We start by introducing the following kernel:

$$K^{\pm}(x,y) = \exp(\mp\frac{i}{2}\sum_{j=1}^{n}(\grave{x}_{2j-1}\grave{y}_{2j} - \grave{x}_{2j}\grave{y}_{2j-1})) \tag{7.1}$$

where we assume that $\grave{x}_i\grave{y}_j = -\grave{y}_j\grave{x}_i$, i.e. the $\grave{x}_i$ and the $\grave{y}_j$ together generate a Grassmann algebra Λ_{4n} of dimension 2^{4n}.

Easy calculations show that

$$\begin{array}{rcl}
\partial_{\grave{x}_{2i}} K^{\pm}(x,y) & = & \pm\frac{i}{2}\grave{y}_{2i-1}K^{\pm}(x,y)\\
\partial_{\grave{x}_{2i-1}} K^{\pm}(x,y) & = & \mp\frac{i}{2}\grave{y}_{2i}K^{\pm}(x,y)\\
\partial_{\grave{y}_{2i}} K^{\pm}(x,y) & = & \pm\frac{i}{2}\grave{x}_{2i-1}K^{\pm}(x,y)\\
\partial_{\grave{y}_{2i-1}} K^{\pm}(x,y) & = & \mp\frac{i}{2}\grave{x}_{2i}K^{\pm}(x,y).
\end{array}$$

Using the kernel (7.1) we define the fermionic Fourier transform on the Grassmann algebra Λ_{2n}, generated by the $\grave{x}_i$, as

$$\mathcal{F}^{\pm}_{0|2n}(\cdot)(y) = (2\pi)^n \int_{B,x} K^{\pm}(x,y)\,(\cdot). \tag{7.2}$$

Remark 24. *Note that the kernel $K^{\pm}(x,y)$ is symmetric:*

$$K^{\pm}(x,y) = K^{\pm}(y,x).$$

This is clearly not the case if one would use the kernel

$$\exp(\mp i\sum_{j=1}^{2n}\grave{x}_j\grave{y}_j)$$

as is done in e.g. [84, 47, 66]. Moreover, our definition is invariant under symplectic changes of variables, whereas the other approach is invariant under the orthogonal group.

Now we have the following basic lemma concerning the action of variables and derivatives.

Lemma 17. *If $g \in \Lambda_{2n}$, then one has*

$$\begin{array}{rcl}
\mathcal{F}^{\pm}_{0|2n}(\partial_{\grave{x}_{2i}} g) & = & \mp \frac{i}{2} \grave{y}_{2i-1} \mathcal{F}^{\pm}_{0|2n}(g)\\
\mathcal{F}^{\pm}_{0|2n}(\partial_{\grave{x}_{2i-1}} g) & = & \pm \frac{i}{2} \grave{y}_{2i} \mathcal{F}^{\pm}_{0|2n}(g)\\
\mathcal{F}^{\pm}_{0|2n}(\grave{x}_{2i} g) & = & \pm 2i \, \partial_{\grave{y}_{2i-1}} \mathcal{F}^{\pm}_{0|2n}(g)\\
\mathcal{F}^{\pm}_{0|2n}(\grave{x}_{2i-1} g) & = & \mp 2i \, \partial_{\grave{y}_{2i}} \mathcal{F}^{\pm}_{0|2n}(g).
\end{array}$$

Proof. We only prove the first relation, the other proofs being completely similar:

$$\begin{array}{rcl}
\mathcal{F}^{\pm}_{0|2n}(\partial_{\grave{x}_{2i}} g) & = & (2\pi)^n \displaystyle\int_{B,x} K^{\pm}(x,y)(\partial_{\grave{x}_{2i}} g)\\
& = & (2\pi)^n \displaystyle\int_{B,x} \partial_{\grave{x}_{2i}} \left[K^{\pm}(x,y) g\right] - (2\pi)^n \displaystyle\int_{B,x} \left[\partial_{\grave{x}_{2i}} K^{\pm}(x,y)\right] g\\
& = & -(2\pi)^n \displaystyle\int_{B,x} \left[\partial_{\grave{x}_{2i}} K^{\pm}(x,y)\right] g\\
& = & \mp \dfrac{i}{2} \grave{y}_{2i-1} (2\pi)^n \displaystyle\int_{B,x} K^{\pm}(x,y) g\\
& = & \mp \dfrac{i}{2} \grave{y}_{2i-1} \mathcal{F}^{\pm}_{0|2n}(g).
\end{array}$$

Note that the first term in the second line vanishes because we have

$$(2\pi)^n \int_{B,x} \partial_{\grave{x}_{2i}} \left[K^{\pm}(x,y) g\right] = 2^n \partial_{\grave{x}_{2n}} \dots \partial_{\grave{x}_1} \partial_{\grave{x}_{2i}} \left[K^{\pm}(x,y) g\right] = 0.$$

□

Next we consider the action of the Dirac operator and the vector variable. Using the previous lemma this immediately leads to the following.

Corollary 6. *One has the following relations*

$$\begin{array}{rclcrcl}
\mathcal{F}^{\pm}_{0|2n}(\partial_{\underline{\grave{x}}} g) & = & \pm i \underline{\grave{y}} \mathcal{F}^{\pm}_{0|2n}(g) & \qquad & \mathcal{F}^{\pm}_{0|2n}(\Delta_f g) & = & -\underline{\grave{y}}^2 \mathcal{F}^{\pm}_{0|2n}(g)\\
\mathcal{F}^{\pm}_{0|2n}(\underline{\grave{x}} g) & = & \pm i \partial_{\underline{\grave{y}}} \mathcal{F}^{\pm}_{0|2n}(g) & \qquad & \mathcal{F}^{\pm}_{0|2n}(\underline{\grave{x}}^2 g) & = & -\Delta_f \mathcal{F}^{\pm}_{0|2n}(g).
\end{array}$$

Now let us calculate the fermionic Fourier transform of some simple functions:

(i) $\mathcal{F}^{\pm}_{0|2n}(\underline{\grave{x}}^{2n}) = n!\mathcal{F}^{\pm}_{0|2n}(\grave{x}_1 \ldots \grave{x}_{2n}) = n!2^n$.

(ii) $\mathcal{F}^{\pm}_{0|2n}(\underline{\grave{x}}^{2k})$.

We first need the following formula

$$\begin{aligned}
\partial_{\underline{\grave{x}}}^{2n-2k}\underline{\grave{x}}^{2n} &= 2n(2n-2-2n)\partial_{\underline{\grave{x}}}^{2n-2k-2}\underline{\grave{x}}^{2n-2}\\
&= \ldots\\
&= 2^{2n-2k}(-1)^{n-k}\frac{n!(n-k!)}{k!}\underline{\grave{x}}^{2k}.
\end{aligned}$$

We then obtain the following result

$$\begin{aligned}
\mathcal{F}^{\pm}_{0|2n}(\underline{\grave{x}}^{2k}) &= \frac{(-1)^{n-k}}{2^{2n-2k}}\frac{k!}{n!(n-k)!}\mathcal{F}^{\pm}_{0|2n}(\partial_{\underline{\grave{x}}}^{2n-2k}\underline{\grave{x}}^{2n})\\
&= \frac{1}{2^{2n-2k}}\frac{k!}{n!(n-k)!}\underline{\grave{y}}^{2n-2k}\mathcal{F}^{\pm}_{0|2n}(\underline{\grave{x}}^{2n})\\
&= 2^{2k-n}\frac{k!}{(n-k)!}\underline{\grave{y}}^{2n-2k}.
\end{aligned}$$

(iii) The fermionic Fourier transform of the Gaussian $\exp(\underline{\grave{x}}^2/2)$:

$$\begin{aligned}
\mathcal{F}^{\pm}_{0|2n}(\exp(\underline{\grave{x}}^2/2)) &= \sum_{j=0}^{n}\mathcal{F}^{\pm}_{0|2n}(\frac{\underline{\grave{x}}^{2j}}{2^j j!})\\
&= \sum_{j=0}^{n}\frac{2^{2j-n}}{2^j j!}\frac{j!}{(n-j)!}\underline{\grave{y}}^{2n-2j}\\
&= \sum_{j=0}^{n}\frac{\underline{\grave{y}}^{2n-2j}}{2^{n-j}(n-j)!}\\
&= \exp(\underline{\grave{y}}^2/2).
\end{aligned}$$

So we conclude that the Gaussian function is invariant under the fermionic Fourier transform (as would be expected).

Now we turn our attention to the inversion of the Fourier transform. We have the following theorem.

Theorem 42 (Inversion)**.** *One has that*

$$\mathcal{F}^{\pm}_{0|2n} \circ \mathcal{F}^{\mp}_{0|2n} = id_{\Lambda_{2n}}.$$

Proof. As the Fourier transform is linear, it suffices to give the proof for a monomial $x_A = \grave{x}_1^{\alpha_1} \ldots \grave{x}_{2n}^{\alpha_{2n}}$ with $\alpha_i \in \{0,1\}$. We then obtain, using lemma 17:

$$\begin{aligned}
& \mathcal{F}^{+}_{0|2n}\mathcal{F}^{-}_{0|2n}(x_A) \\
= \;& (2i)^{|A|}(-1)^{\sum \alpha_{2i}} \mathcal{F}^{+}_{0|2n}\left(\partial_{\grave{y}_2}^{\alpha_1}\partial_{\grave{y}_1}^{\alpha_2} \ldots \partial_{\grave{y}_{2n}}^{\alpha_{2n-1}}\partial_{\grave{y}_{2n-1}}^{\alpha_{2n}} \mathcal{F}^{-}_{0|2n}(1)\right) \\
= \;& (2i)^{|A|}(-1)^{\sum \alpha_{2i}} (\frac{i}{2})^{|A|}(-1)^{\sum \alpha_{2i-1}} \grave{x}_1^{\alpha_1} \ldots \grave{x}_{2n}^{\alpha_{2n}} \mathcal{F}^{+}_{0|2n}\mathcal{F}^{-}_{0|2n}(1) \\
= \;& x_A \mathcal{F}^{+}_{0|2n}(\frac{2^{-n}}{n!}\underline{\grave{y}}^{2n}) \\
= \;& x_A.
\end{aligned}$$

Similarly we find that $\mathcal{F}^{-}_{0|2n}\mathcal{F}^{+}_{0|2n}(x_A) = x_A$. □

As a consequence we immediately obtain that $\mathcal{F}^{\pm}_{0|2n}$ is an isomorphism of Λ_{2n}.

Now we take a first important step in the construction of an eigenbasis of the Fourier transform.

Theorem 43. *Let $H_l(x) \in \mathcal{H}_l^f$ be a spherical harmonic of degree l. Then one has*

$$\mathcal{F}^{\pm}_{0|2n}(H_l(x)\exp(\underline{\grave{x}}^2/2))(y) = (\pm i)^l H_l(y)\exp(\underline{\grave{y}}^2/2).$$

Proof. In a similar way as in theorem 34, we can write

$$\int_{B,x} \exp(\underline{\grave{x}}^2/2) R = \sum_{k=0}^{n} \frac{(-1)^k (2\pi)^{-n}}{2^k k!} (\Delta_f^k R)(0)$$

where $R \in \Lambda_{2n}$. Using this expression we find

$$\begin{aligned}
& \mathcal{F}^{\pm}_{0|2n}(H_l(x)\exp(\underline{\grave{x}}^2/2)) \\
= \;& \sum_{k=0}^{n} \frac{(-1)^k}{2^k k!} \Delta_f^k \left(K^{\pm}(x,y) H_l(x)\right)(0) \\
= \;& \sum_{k=0}^{n}\sum_{j=0}^{2n} \frac{(-1)^k (\pm i)^j}{2^k k! j!} \Delta_f^k \left((-1)^j \langle \underline{\grave{x}}, \underline{\grave{y}}\rangle^j H_l(x)\right)(0) \\
= \;& \sum_{k \geq l/2}^{n} \frac{(-1)^k (\pm i)^{2k-l}}{2^k k! (2k-l)!} \Delta_f^k \left((-1)^{2k-l} \langle \underline{\grave{x}}, \underline{\grave{y}}\rangle^{2k-l} H_l(x)\right)
\end{aligned}$$

with $\langle \underline{\grave{x}}, \underline{\grave{y}} \rangle = \sum_{j=1}^{n} (\grave{x}_{2j-1}\grave{y}_{2j} - \grave{x}_{2j}\grave{y}_{2j-1})/2$.

In a similar way as in theorem 32 (Funk-Hecke), we can prove that the following relation holds:

$$\Delta_f^s (-1)^k \langle \underline{\grave{x}}, \underline{\grave{y}} \rangle^k H_l(x) = \frac{2^{2s} s! n!}{(n-s)!} \alpha_l^*(t^k) H_l(y) \underline{\grave{y}}^{k-l} (-1)^{(k-l)/2}$$

with $k + l = 2s \leq 2n$ and with

$$\begin{aligned}
\alpha_l^*(t^k) &= \frac{k!(n-\frac{k+l}{2})!}{(k-l)!n!} \frac{\pi^{-\frac{1}{2}}(-1)^{\frac{k+l}{2}}}{2^l} \Gamma(\frac{k-l+1}{2}) \quad \text{if } k+l \text{ even}, \ k \geq l \\
&= 0 \quad \text{if } k+l \text{ odd} \\
&= 0 \quad \text{if } k < l.
\end{aligned}$$

We then obtain

$$\begin{aligned}
& \sum_{k \geq l/2}^{n} \frac{(-1)^k (\pm i)^{2k-l}}{2^k k! (2k-l)!} \Delta_f^k \left((-1)^{2k-l} \langle \underline{\grave{x}}, \underline{\grave{y}} \rangle^{2k-l} H_l(x) \right) \\
= & \sum_{p=0}^{n-l} \frac{(-1)^{l+p} (\pm i)^{l+2p}}{2^{l+p} (l+p)! (l+2p)!} \Delta_f^{l+p} \left((-1)^{l+2p} \langle \underline{\grave{x}}, \underline{\grave{y}} \rangle^{l+2p} H_l(x) \right) \\
= & \sum_{p=0}^{n-l} \frac{(-1)^{l+p} (\pm i)^{l+2p}}{2^{l+p} (l+p)! (l+2p)!} \frac{2^{2l+2p} (l+p)! n!}{(n-l-p)!} \alpha_l^*(t^{l+2p}) H_l(y) \underline{\grave{y}}^{2p} (-1)^p \\
= & \sum_{p=0}^{n-l} \frac{(-1)^{l+p} (\pm i)^{l+2p}}{(l+2p)!} \frac{2^{l+p} n!}{(n-l-p)!} \frac{(l+2p)!(n-l-p)!}{(2p)! n!} \\
& \times \frac{\pi^{-\frac{1}{2}} (-1)^{l+p}}{2^l} \Gamma(\frac{2p+1}{2}) H_l(y) \underline{\grave{y}}^{2p} (-1)^p \\
= & (\pm i)^l \sum_{p=0}^{n-l} \frac{\underline{\grave{y}}^{2p}}{2^p p!} H_l(y) \\
= & (\pm i)^l \sum_{p=0}^{n} \frac{\underline{\grave{y}}^{2p}}{2^p p!} H_l(y) \\
= & (\pm i)^l H_l(y) \exp(\underline{\grave{y}}^2/2),
\end{aligned}$$

where we have used the fact that $\underline{\grave{y}}^{2k} H_l(y) = 0$ if $k > n - l$. $\square$

As a consequence, we obtain the following theorem, which completely characterizes the fermionic Fourier transform because of the Fischer decomposition of Λ_{2n} (see theorem 4):

$$\Lambda_{2n} = \bigoplus_{k=0}^{n} \left(\bigoplus_{j=0}^{n-k} \underline{x}\grave{}^{2j} \mathcal{H}_k^f \right). \tag{7.3}$$

Theorem 44. *One has that*

$$\mathcal{F}^{\pm}_{0|2n}(\underline{x}\grave{}^{2k}\mathcal{H}_l^f) = (\pm i)^l \frac{2^{2k+l-n}k!}{(n-k-l)!} \underline{y}\grave{}^{2n-2k-2l}\mathcal{H}_l^f$$

with $l \leq n$ and $k \leq n-l$.

Proof. Immediately by using theorem 43 and noting that $\mathcal{F}^{\pm}_{0|2n}$ maps k-homogeneous elements of Λ_{2n} to $(2n-k)$-homogeneous elements. □

We also have a Parseval theorem for the fermionic Fourier transform.

Theorem 45 (Parseval)**.** *Let $f, g \in \Lambda_{2n}$. Then one has:*

$$\int_{B,x} f(x)\overline{g(x)} = \int_{B,y} \mathcal{F}^{\pm}_{0|2n}(f)(y)\overline{\mathcal{F}^{\pm}_{0|2n}(g)(y)}$$

where the bar denotes standard complex conjugation.

Proof. The right-hand side is calculated as follows:

$$\begin{aligned}
RH &= (2\pi)^{2n} \int_{B,y} \int_{B,u} \int_{B,v} e^{\mp i \langle \underline{y}\grave{}, \underline{u}\grave{} \rangle} e^{\pm i \langle \underline{y}\grave{}, \underline{v}\grave{} \rangle} f(u)\overline{g(v)} \\
&= (2\pi)^{n} \int_{B,u} \int_{B,v} \left[(2\pi)^n \int_{B,y} e^{\mp i \langle \underline{y}\grave{}, \underline{u}\grave{} - \underline{v}\grave{} \rangle} \right] f(u)\overline{g(v)} \\
&= (2\pi)^{n} \int_{B,u} \int_{B,v} \frac{2^{-n}}{n!} (\underline{u}\grave{} - \underline{v}\grave{})^{2n} f(u)\overline{g(v)} \\
&= \int_{B,u} \int_{B,v} \delta(\underline{u}\grave{} - \underline{v}\grave{}) f(u)\overline{g(v)} \\
&= \int_{B,u} f(u)\overline{g(u)}.
\end{aligned}$$

□

7.2 The general Fourier transform

We can define a Fourier transform on the whole superspace by

$$\mathcal{F}^{\pm}_{m|2n} = (2\pi)^{-\frac{M}{2}} \int_{\mathbb{R}^m} dV(\underline{x}) \int_{B,x} \exp\left(\mp i \langle x, y \rangle\right)$$

with

$$\langle x, y \rangle = -\sum_{i=1}^{m} x_i y_i + \frac{1}{2} \sum_{j=1}^{n} (\grave{x}_{2j-1} \grave{y}_{2j} - \grave{x}_{2j} \grave{y}_{2j-1}).$$

Note that the kernel $\langle x, y \rangle$ is obtained by taking the anti-commutator of two vector variables x and y as in formula (2.2).

We immediately observe that

$$\mathcal{F}^{\pm}_{m|2n} = \mathcal{F}^{\pm}_{m|0} \circ \mathcal{F}^{\pm}_{0|2n} = \mathcal{F}^{\pm}_{0|2n} \circ \mathcal{F}^{\pm}_{m|0} \tag{7.4}$$

which allows us to use the results of the previous section to study the general Fourier transform. The inversion of the transform is given in the following theorem.

Theorem 46 (Inversion)**.** *One has that*

$$\mathcal{F}^{\pm}_{m|2n} \circ \mathcal{F}^{\mp}_{m|2n} = id_{\mathcal{S}(\mathbb{R}^m)_{m|2n}}$$

and consequently that $\mathcal{F}^{\pm}_{m|2n}$ is an isomorphism of $\mathcal{S}(\mathbb{R}^m)_{m|2n}$.

Proof. We have the following calculation

$$\begin{aligned}
\mathcal{F}^{+}_{m|2n} \circ \mathcal{F}^{-}_{m|2n} &= \mathcal{F}^{+}_{m|0} \circ \mathcal{F}^{+}_{0|2n} \circ \mathcal{F}^{-}_{0|2n} \circ \mathcal{F}^{-}_{m|0} \\
&= \mathcal{F}^{+}_{m|0} \circ \mathrm{id}_{\Lambda_{2n}} \circ \mathcal{F}^{-}_{m|0} \\
&= \mathrm{id}_{\mathcal{S}(\mathbb{R}^m)_{m|2n}}.
\end{aligned}$$

□

The action of derivatives and variables is summarized in the following lemma.

Lemma 18. *If $g \in \mathcal{S}(\mathbb{R}^m)_{m|2n}$, then the following relations hold:*

$$\begin{aligned}
\mathcal{F}^{\pm}_{m|2n}(\partial_{x_i} g) &= \mp i y_i \mathcal{F}^{\pm}_{m|2n}(g) \\
\mathcal{F}^{\pm}_{m|2n}(\partial_{\grave{x}_{2i}} g) &= \mp \tfrac{i}{2} \grave{y}_{2i-1} \mathcal{F}^{\pm}_{m|2n}(g) \\
\mathcal{F}^{\pm}_{m|2n}(\partial_{\grave{x}_{2i-1}} g) &= \pm \tfrac{i}{2} \grave{y}_{2i} \mathcal{F}^{\pm}_{m|2n}(g)
\end{aligned}$$

$$\begin{aligned}
\mathcal{F}^{\pm}_{m|2n}(x_i g) &= \mp i\partial_{y_i}\mathcal{F}^{\pm}_{m|2n}(g)\\
\mathcal{F}^{\pm}_{m|2n}(\grave{x}_{2i} g) &= \pm 2i\ \partial_{\grave{y}_{2i-1}}\mathcal{F}^{\pm}_{m|2n}(g)\\
\mathcal{F}^{\pm}_{m|2n}(\grave{x}_{2i-1} g) &= \mp 2i\ \partial_{\grave{y}_{2i}}\mathcal{F}^{\pm}_{m|2n}(g).
\end{aligned}$$

The action of the Dirac operator and the vector variable is given by

$$\begin{aligned}
\mathcal{F}^{\pm}_{m|2n}(\partial_x g) &= \pm iy\mathcal{F}^{\pm}_{m|2n}(g) & \qquad \mathcal{F}^{\pm}_{m|2n}(\Delta g) &= -y^2\mathcal{F}^{\pm}_{m|2n}(g)\\
\mathcal{F}^{\pm}_{m|2n}(x g) &= \pm i\partial_y\mathcal{F}^{\pm}_{m|2n}(g) & \qquad \mathcal{F}^{\pm}_{m|2n}(x^2 g) &= -\partial_y^2\mathcal{F}^{\pm}_{m|2n}(g).
\end{aligned}$$

Proof. Immediate, using formula (7.4), lemma 17 and the basic properties of the bosonic Fourier transform. □

Corollary 7. *The Lie algebra $\mathfrak{sl}_2(\mathbb{R})$, generated by Δ, x^2 and $2\mathbb{E}+M$ is invariant under the action of $\mathcal{F}^{\pm}_{m|2n}$.*

Proof. We already have that $\mathcal{F}^{\pm}_{m|2n}(\Delta g) = -y^2\mathcal{F}^{\pm}_{m|2n}(g)$ and $\mathcal{F}^{\pm}_{m|2n}(x^2 g) = -\partial_y^2\mathcal{F}^{\pm}_{m|2n}(g)$. As $\left[\Delta, x^2\right] = 2(2\mathbb{E}+M)$ we obtain

$$\begin{aligned}
\mathcal{F}^{\pm}_{m|2n}((2\mathbb{E}+M)g) &= \frac{1}{2}\mathcal{F}^{\pm}_{m|2n}(\left[\Delta, x^2\right] g)\\
&= -\frac{1}{2}\left[\partial_y^2, y^2\right]\mathcal{F}^{\pm}_{m|2n}(g)\\
&= -(2\mathbb{E}_y+M)\mathcal{F}^{\pm}_{m|2n}(g),
\end{aligned}$$

completing the proof. □

We also have the following lemma concerning the Fourier transform of the convolution of two functions.

Lemma 19. *Let f, g be elements of $L_1(\mathbb{R}^m)_{m|2n}$. Then the following holds*

$$\mathcal{F}^{\pm}_{m|2n}(f*g) = (2\pi)^{M/2}\mathcal{F}^{\pm}_{m|2n}(f)\mathcal{F}^{\pm}_{m|2n}(g).$$

Proof. We calculate the left-hand side as

$$\begin{aligned}
&(2\pi)^{M/2}\mathcal{F}^{\pm}_{m|2n}(f*g)\\
=\ & \int_{\mathbb{R}^{m|2n},u}\int_{\mathbb{R}^{m|2n},x}\exp\left(\mp i\langle u,y\rangle\right)f(u-x)g(x)\\
=\ & \int_{\mathbb{R}^{m|2n},x}\left[\int_{\mathbb{R}^{m|2n},u}\exp\left(\mp i\langle u-x,y\rangle\right)f(u-x)\right]\exp\left(\mp i\langle x,y\rangle\right)g(x)
\end{aligned}$$

$$\begin{aligned} &= \int_{\mathbb{R}^{m|2n},x} (2\pi)^{M/2} \mathcal{F}^{\pm}_{m|2n}(f)(y) \exp\left(\mp i\langle x,y\rangle\right) g(x) \\ &= (2\pi)^{M} \mathcal{F}^{\pm}_{m|2n}(f)(y) \mathcal{F}^{\pm}_{m|2n}(g)(y) \end{aligned}$$

which completes the proof. □

Next we have the following full Parseval theorem.

Theorem 47 (Parseval). *Let $f, g \in L_2(\mathbb{R}^m)_{m|2n}$. Then the following holds:*

$$\int_{\mathbb{R}^{m|2n},x} f(x)\overline{g(x)} = \int_{\mathbb{R}^{m|2n},y} \mathcal{F}^{\pm}_{m|2n}(f)(y)\overline{\mathcal{F}^{\pm}_{m|2n}(g)(y)}.$$

Proof. This follows immediately from the classical Parseval theorem combined with theorem 45. □

Let us now calculate the Fourier transform of the super Gaussian function:

$$\begin{aligned} \mathcal{F}^{\pm}_{m|2n}(\exp(x^2/2)) &= \mathcal{F}^{\pm}_{m|2n}(\exp(\underline{x}^2/2)\exp(\underline{x \grave{}}^2/2)) \\ &= \mathcal{F}^{\pm}_{m|0}(\exp(\underline{x}^2/2))\mathcal{F}^{\pm}_{0|2n}(\exp(\underline{x \grave{}}^2/2)) \\ &= \exp(\underline{y}^2/2)\exp(\underline{y \grave{}}^2/2) \\ &= \exp(y^2/2). \end{aligned}$$

The theorems 41 and 43 can be merged into the following theorem. The proof makes extensive use of the Clifford-Hermite polynomials introduced in chapter 4.

Theorem 48. *Let $H_l(x) \in \mathcal{H}_l$ be a spherical harmonic of degree l. Then one has*

$$\mathcal{F}^{\pm}_{m|2n}(H_l(x)\exp(x^2/2))(y) = (\pm i)^l H_l(y)\exp(y^2/2).$$

Proof. Due to theorem 8 it suffices to give the proof in the case where $H_l(x)$ is of the following form:

$$H_l(x) = f_{k,l-2k-j,j} H^b_{l-2k-j} H^f_j$$

with

$$f_{k,l-2k-j,j}(\underline{x}^2, \underline{x \grave{}}^2) = \sum_{i=0}^{k} \binom{k}{i} \frac{(n-j-i)!}{\Gamma(\frac{m}{2}+l-k-j-i)} \underline{x}^{2k-2i} \underline{x \grave{}}^{2i}$$

and with $H^b_{l-2k-j} \in \mathcal{H}^b_{l-2k-j}$, $H^f_j \in \mathcal{H}^f_j$.

We then calculate the Fourier transform of $H_l(x)\exp(x^2/2)$ as

$$
\begin{aligned}
& \mathcal{F}^{\pm}_{m|2n}(H_l(x)\exp(x^2/2)) \\
= \; & \mathcal{F}^{\pm}_{m|2n}(f_{k,l-2k-j,j} H^b_{l-2k-j} H^f_j \exp(x^2/2)) \\
= \; & (-1)^k \sum_{i=0}^{k} \binom{k}{i} \frac{(n-j-i)!}{\Gamma(\frac{m}{2}+l-k-j-i)} \\
& \times \partial_{\underline{y}}^{2k-2i} \partial_{\grave{\underline{y}}}^{2i} \mathcal{F}^{\pm}_{m|2n}(H^b_{l-2k-j} H^f_j \exp(x^2/2)) \\
= \; & (-1)^k \sum_{i=0}^{k} \binom{k}{i} \frac{(n-j-i)!}{\Gamma(\frac{m}{2}+l-k-j-i)} \\
& \times \partial_{\underline{y}}^{2k-2i} \partial_{\grave{\underline{y}}}^{2i} \mathcal{F}^{\pm}_{m|0}(H^b_{l-2k-j} \exp(\underline{x}^2/2)) \mathcal{F}^{\pm}_{0|2n}(H^f_j \exp(\grave{\underline{x}}^2/2)) \\
= \; & (\pm i)^l \sum_{i=0}^{k} \binom{k}{i} \frac{(n-j-i)!}{\Gamma(\frac{m}{2}+l-k-j-i)} \\
& \times \partial_{\underline{y}}^{2k-2i} \partial_{\grave{\underline{y}}}^{2i} H^b_{l-2k-j}(\underline{y}) \exp(\underline{y}^2/2) H^f_j(\grave{\underline{y}}) \exp(\grave{\underline{y}}^2/2) \\
= \; & (\pm i)^l \sum_{i=0}^{k} \binom{k}{i} \frac{(n-j-i)!}{\Gamma(\frac{m}{2}+l-k-j-i)} \widetilde{CH}_{2k-2i,m,l-2k-j}(\underline{y}) \widetilde{CH}_{2i,-2n,j}(\grave{\underline{y}}) \\
& \times H^b_{l-2k-j}(\underline{y}) \exp(\underline{y}^2/2) H^f_j(\grave{\underline{y}}) \exp(\grave{\underline{y}}^2/2).
\end{aligned}
$$

So we still need to prove that

$$
\sum_{i=0}^{k} \binom{k}{i} \frac{(n-j-i)!}{\Gamma(\frac{m}{2}+l-k-j-i)} \widetilde{CH}_{2k-2i,m,l-2k-j}(\underline{y}) \widetilde{CH}_{2i,-2n,j}(\grave{\underline{y}}) \tag{7.5}
$$

equals $f_{k,l-2k-j,j}(\underline{y}^2, \grave{\underline{y}}^2)$. The explicit form of the rescaled Clifford-Hermite polynomials (see formula (4.4)) is given by

$$
\widetilde{CH}_{2t,M,k}(y) = \sum_{i=0}^{t} 2^{2t-2i} \left(\begin{array}{c} t \\ i \end{array} \right) \frac{\Gamma(t+k+M/2)}{\Gamma(i+k+M/2)} y^{2i}
$$

or, in the case where $M = -2n$ and $m = 0$, by

$$
\widetilde{CH}_{2t,-2n,k}(\grave{\underline{y}}) = \sum_{i=0}^{t} (-1)^{t-i} 2^{2t-2i} \left(\begin{array}{c} t \\ i \end{array} \right) \frac{(n-k-i)!}{(n-k-t)!} \grave{\underline{y}}^{2i}.
$$

Plugging these expressions into formula (7.5) yields

$$
\begin{aligned}
& \sum_{i=0}^{k} \binom{k}{i} \frac{(n-j-i)!}{\Gamma(\frac{m}{2}+l-k-j-i)} \widetilde{CH}_{2k-2i,m,l-2k-j}(\underline{y}) \widetilde{CH}_{2i,-2n,j}(\underline{\grave{y}}) \\
= & \sum_{i=0}^{k} \sum_{p=0}^{k-i} \sum_{q=0}^{i} 2^{2k-2p-2q} (-1)^{i-q} \binom{k}{i} \binom{k-i}{p} \binom{i}{q} \\
& \times \frac{(n-j-q)!}{\Gamma(\frac{m}{2}+p+l-2k-j)} \underline{y}^{2p} \underline{\grave{y}}^{2q} \\
= & \sum_{p=0}^{k} \sum_{q=0}^{k-p} \sum_{i=q}^{k-p} 2^{2k-2p-2q} (-1)^{i-q} \binom{k}{i} \binom{k-i}{p} \binom{i}{q} \\
& \times \frac{(n-j-q)!}{\Gamma(\frac{m}{2}+p+l-2k-j)} \underline{y}^{2p} \underline{\grave{y}}^{2q} \\
= & \sum_{p=0}^{k} \sum_{q=0}^{k-p} 2^{2k-2p-2q} (-1)^{q} \frac{(n-j-q)!}{\Gamma(\frac{m}{2}+p+l-2k-j)} \\
& \times \underline{y}^{2p} \underline{\grave{y}}^{2q} \sum_{i=q}^{k-p} (-1)^{i} \binom{k}{i} \binom{k-i}{p} \binom{i}{q} \\
= & \sum_{p=0}^{k} \binom{k}{p} \frac{(n-j-k+p)!}{\Gamma(\frac{m}{2}+p+l-2k-j)} \underline{y}^{2p} \underline{\grave{y}}^{2k-2p} \\
= & \sum_{i=0}^{k} \binom{k}{i} \frac{(n-j-i)!}{\Gamma(\frac{m}{2}+l-k-j-i)} \underline{y}^{2k-2i} \underline{\grave{y}}^{2i} \\
= & f_{k,l-2k-j,j}(\underline{y}^2, \underline{\grave{y}}^2)
\end{aligned}
$$

since

$$
\sum_{i=q}^{k-p} (-1)^{i} \binom{k}{i} \binom{k-i}{p} \binom{i}{q} = \begin{cases} (-1)^q \binom{k}{p} & \text{if } k = p+q \\ 0 & \text{otherwise.} \end{cases}
$$

This completes the proof of the theorem. □

As a consequence we immediately obtain

Corollary 8. *If $H_l(x) \in \mathcal{H}_l$, then the following holds*

$$
H_l(\partial_x) \exp(x^2/2) = H_l(x) \exp(x^2/2)
$$

where $H_l(\partial_x)$ is the operator obtained by substituting $x_i \rightarrow -\partial_{x_i}$, $\grave{x}_{2i} \rightarrow 2\partial_{\grave{x}_{2i-1}}$ and $\grave{x}_{2i-1} \rightarrow -2\partial_{\grave{x}_{2i}}$ in $H_l(x)$.

Under mild assumptions on the super-dimension M we can prove the converse of the previous corollary.

Lemma 20. *Suppose $M \notin -2\mathbb{N}$ or $m = 0$. If $p(x)$ is a homogeneous polynomial of degree k satisfying*

$$p(\partial_x) \exp(x^2/2) = p(x) \exp(x^2/2)$$

then $p(x) \in \mathcal{H}_k$.

Proof. As $M \notin -2\mathbb{N}$ or $m = 0$, p can be developed in a Fischer decomposition (see theorem 4):

$$p = \sum_{j=0}^{\lfloor k/2 \rfloor} x^{2j} H_{k-2j}$$

with $H_{k-2j} \in \mathcal{H}_{k-2j}$. We then have that

$$\begin{aligned} p(\partial_x) \exp(x^2/2) &= \sum_{j=0}^{\lfloor k/2 \rfloor} \Delta^j H_{k-2j}(\partial_x) \exp(x^2/2) \\ &= \sum_{j=0}^{\lfloor k/2 \rfloor} \Delta^j H_{k-2j}(x) \exp(x^2/2) \\ &= \sum_{j=0}^{\lfloor k/2 \rfloor} \widetilde{CH}_{2j,M,k-2j}(x) H_{k-2j}(x) \exp(x^2/2) \end{aligned}$$

where we have used corollary 8. From this expression it is clear that

$$p(\partial_x) \exp(x^2/2) = p(x) \exp(x^2/2)$$

if and only if $p(x) = H_k(x)$. □

Now we can construct an eigenfunction basis of the super Fourier transform. We restrict ourselves for the rest of this section to the case $M \notin -2\mathbb{N}$ or $m = 0$. We then consider, as in section 4.2, the set of functions $\phi_{j,k,l}(x)$ defined by

$$\phi_{j,k,l}(x) = (\partial_x + x)^j M_k^{(l)} \exp x^2/2 = H_{j,M,k}(x) M_k^{(l)} \exp x^2/2$$

which form a basis for $\mathcal{S}(\mathbb{R}^m)_{m|2n}$. If we calculate the Fourier transform of these functions, we obtain

$$\begin{aligned}
\mathcal{F}^{\pm}_{m|2n}(\phi_{j,k,l}(x)) &= \mathcal{F}^{\pm}_{m|2n}((\partial_x + x)^j M_k^{(l)} \exp x^2/2) \\
&= (\pm i)^j (\partial_y + y)^j \mathcal{F}^{\pm}_{m|2n}(M_k^{(l)} \exp x^2/2) \\
&= (\pm i)^j (\pm i)^k (\partial_y + y)^j M_k^{(l)} \exp y^2/2 \\
&= (\pm i)^{j+k} \phi_{j,k,l}(y) \\
&= e^{\pm i (j+k)\frac{\pi}{2}} \phi_{j,k,l}(y)
\end{aligned}$$

where we have used theorem 48 and lemma 18. This means that the Fourier transform rotates the basic functions over a multiple of a right angle.

Similarly, we obtain that the Fourier transform acts on the scalar basis $\psi_{j,k,l}(x)$ (see again section 4.2) as

$$\mathcal{F}^{\pm}_{m|2n}(\psi_{j,k,l}(x)) = e^{\pm i (2j+k)\frac{\pi}{2}} \psi_{j,k,l}(y).$$

On the other hand we have that the functions $\phi_{j,k,l}(x)$ are solutions of a super harmonic oscillator (see section 4.4). This means that they satisfy

$$\frac{1}{2}(\Delta - x^2)\phi_{j,k,l}(x) = \left(\frac{M}{2} + (j+k)\right)\phi_{j,k,l}(x).$$

This implies that the operator exponential $\mathcal{K}$ defined by

$$\mathcal{K}^{\pm} = \exp \pm \frac{i\pi}{4}(\Delta - x^2 - M)$$

satisfies

$$\mathcal{K}^{\pm}\phi_{j,k,l}(x) = e^{\pm i (j+k)\frac{\pi}{2}} \phi_{j,k,l}(x)$$

and thus equals the Fourier transform. In this way we have proven the following theorem.

Theorem 49. *The super Fourier transform can equivalently be defined by the operator exponential*

$$\mathcal{F}^{\pm}_{m|2n} = \exp \pm \frac{i\pi}{4}(\Delta - x^2 - M).$$

Remark 25. *It is also possible to define a Fourier transform in the framework of Dunkl operators (see [49, 34]), using a suitable generalized exponential. In that case, an eigenfunction basis of this so-called Dunkl-transform is also given by the Clifford-Hermite functions (see [86]).*

7.3 Fundamental solution of the super Laplace operator

We do not aim at developing in this section a complete theory of the Fourier transform on distributions in superspace. We restrict ourselves to what is necessary for establishing the fundamental solution of the super Laplace operator Δ.

We first calculate the Fourier transform of the super Dirac distribution. This yields

$$\begin{aligned}
\mathcal{F}^{\pm}_{m|2n}(\delta(x)) &= \mathcal{F}^{\pm}_{m|0}(\delta(\underline{x}))\mathcal{F}^{\pm}_{0|2n}(\frac{\pi^n}{n!}\underline{\grave{x}}^{2n})\\
&= (2\pi)^{-M/2}.
\end{aligned}$$

Fourier transformation of the super Poisson equation $\Delta f(x) = \delta(x)$ then yields the algebraic equation

$$-y^2 F(y) = (2\pi)^{-M/2}$$

with $F(y) = \mathcal{F}^{\pm}_{m|2n}(f)$. Assuming $m > 0$, we have that

$$\begin{aligned}
(y^2)^{-1} &= \underline{y}^{-2}\left(1 + \frac{\underline{\grave{y}}^2}{\underline{y}^2}\right)^{-1}\\
&= \underline{y}^{-2}\sum_{k=0}^{n}(-1)^k\left(\frac{\underline{\grave{y}}^2}{\underline{y}^2}\right)^k.
\end{aligned}$$

It follows that the fundamental solution of the Laplace operator is given by the inverse Fourier transform of the following distribution

$$F(y) = -(2\pi)^{-M/2}\underline{y}^{-2}\sum_{k=0}^{n}(-1)^k\left(\frac{\underline{\grave{y}}^2}{\underline{y}^2}\right)^k.$$

We obtain that

$$\begin{aligned}
f(x) &= -(2\pi)^{-M/2}\sum_{k=0}^{n}(-1)^k\mathcal{F}^{\mp}_{m|0}(\underline{y}^{-2k-2})\mathcal{F}^{\mp}_{0|2n}(\underline{\grave{y}}^{2k})\\
&= -(2\pi)^{-M/2}\sum_{k=0}^{n}(-1)^k\mathcal{F}^{\mp}_{m|0}(\underline{y}^{-2k-2})2^{2k-n}\frac{k!}{(n-k)!}\underline{\grave{x}}^{2n-2k}\\
&= \pi^n\sum_{k=0}^{n}\frac{2^{2k}k!}{(n-k)!}\mathcal{F}^{\mp}_{m|0}\left(\frac{(2\pi)^{-m/2}}{r^{2k+2}}\right)\underline{\grave{x}}^{2n-2k}
\end{aligned}$$

with $r^2 = -\underline{y}^2$. In this expression, $(2\pi)^{-m/2} r^{-2k-2}$ is nothing else but the Fourier transform in distributional sense of the fundamental solution of the $(k+1)$th power of the classical Laplace operator (see e.g. [57]). Denoting this fundamental solution by $\nu_{2k+2}^{m|0}$ we find that

$$f(x) = \pi^n \sum_{k=0}^{n} \frac{4^k k!}{(n-k)!} \nu_{2k+2}^{m|0} \grave{\underline{x}}^{2n-2k}.$$

This result is the same as the one obtained in theorem 37 using a completely different method. The Fourier method described in this section can also be used to construct a fundamental solution for the operator Δ^l. The advantage of the Fourier technique is that we do not need the rather complicated lemma 16, the drawback is that we can only calculate the fundamental solution of operators that transform nicely under the Fourier transform.

7.4 The fractional Fourier transform

An extension of the fractional Fourier transform (see e.g. [75], the book [77] and references therein) to superspace may be introduced by

$$\mathcal{F}^a_{m|2n} = \exp \frac{ia\pi}{4} (\Delta - x^2 - M),$$

where $a \in [-1, 1]$. The case $a = \pm 1$ is the Fourier transform studied in section 7.2. It is easy to see that this transform acts on the basis $\phi_{j,k,l}(x)$ as follows:

$$\mathcal{F}^a_{m|2n}(\phi_{j,k,l}(x)) = e^{ia(j+k)\frac{\pi}{2}} \phi_{j,k,l}(x).$$

The fractional Fourier transform thus rotates the basic functions over a multiple of the angle $\alpha = a\pi/2$.

The following theorem is easily proven.

Theorem 50. *The fractional Fourier transform satisfies*

$$\mathcal{F}^{a+b}_{m|2n} = \mathcal{F}^a_{m|2n} \circ \mathcal{F}^b_{m|2n} = \mathcal{F}^b_{m|2n} \circ \mathcal{F}^a_{m|2n}.$$

The inverse of the fractional Fourier transform is then immediately obtained:

$$\mathcal{F}^a_{m|2n} \circ \mathcal{F}^{-a}_{m|2n} = \mathrm{id}_{\mathcal{S}(\mathbb{R}^m)_{m|2n}} = \mathcal{F}^{-a}_{m|2n} \circ \mathcal{F}^a_{m|2n}.$$

Now we construct an integral representation of the fractional Fourier transform. In the following lemma we first study the fermionic case with only two variables.

Lemma 21. *The fractional Fourier transform in two anti-commuting variables has the following integral representation:*

$$\mathcal{F}^a_{0|2} = \pi(1-e^{2i\alpha}) \int_{B,x} \exp \frac{2e^{i\alpha}(\grave{y}_2\grave{x}_1 - \grave{y}_1\grave{x}_2) + (1+e^{2i\alpha})(\grave{x}_1\grave{x}_2 + \grave{y}_1\grave{y}_2)}{2-2e^{2i\alpha}} \tag{7.6}$$

with $\alpha = a\pi/2$.

Proof. By definition we have that

$$\mathcal{F}^a_{0|2} = \exp i\alpha H$$

with $\alpha = a\pi/2$ and with $H = 2\partial_{\grave{x}_1}\partial_{\grave{x}_2} - \grave{x}_1\grave{x}_2/2 + 1$. This operator $\mathcal{F}^a_{0|2}$ acts on $\Lambda_2 = \mathrm{span}(1, \grave{x}_1, \grave{x}_2, \grave{x}_1\grave{x}_2)$. Calculating the iterated action of H on this basis yields

$$\begin{array}{lcl} H^k(1) & = & \frac{1}{4}2^k(2-\grave{x}_1\grave{x}_2) \\ H^k(\grave{x}_1) & = & \grave{x}_1 \\ H^k(\grave{x}_2) & = & \grave{x}_2 \\ H^k(\grave{x}_1\grave{x}_2) & = & -\frac{1}{2}2^k(2-\grave{x}_1\grave{x}_2). \end{array}$$

Using these results we find that the fractional Fourier transform acts on this basis as

$$\begin{array}{lcl} \mathcal{F}^a_{0|2}(1) & = & \frac{1}{2}(1+e^{2i\alpha}) + \frac{1}{4}(1-e^{2i\alpha})\grave{x}_1\grave{x}_2 \\ \mathcal{F}^a_{0|2}(\grave{x}_1) & = & e^{i\alpha}\grave{x}_1 \\ \mathcal{F}^a_{0|2}(\grave{x}_2) & = & e^{i\alpha}\grave{x}_2 \\ \mathcal{F}^a_{0|2}(\grave{x}_1\grave{x}_2) & = & (1-e^{2i\alpha}) + \frac{1}{2}(1+e^{2i\alpha})\grave{x}_1\grave{x}_2. \end{array}$$

Requiring the integral operator

$$\int_{B,x} F(x,y)$$

with

$$F(x,y) = \pi\left(f_0(y) + f_1(y)\grave{x}_1 + f_2(y)\grave{x}_2 + f_{12}(y)\grave{x}_1\grave{x}_2\right)$$

and where $f_0(y), f_1(y), f_2(y)$ and $f_{12}(y)$ are still to be determined, to equal $\mathcal{F}^a_{0|2}$ when acting on Λ_2, we obtain

$$\begin{aligned} F(x,y) \quad = \quad & \pi\Bigg((1-e^{2i\alpha}) + \frac{1}{2}(1+e^{2i\alpha})\grave{y}_1\grave{y}_2 + e^{i\alpha}[\grave{y}_2\grave{x}_1 - \grave{y}_1\grave{x}_2] \\ & +[\frac{1}{2}(1+e^{2i\alpha}) + \frac{1}{4}(1-e^{2i\alpha})\grave{y}_1\grave{y}_2]\grave{x}_1\grave{x}_2\Bigg). \end{aligned}$$

It is now easy to check that expanding the kernel given in formula (7.6) yields the same result. $\square$

Using this lemma we obtain the following integral representation of the general fractional Fourier transform.

Theorem 51. *One has that*

$$\mathcal{F}^{a}_{m|2n} = \left(\pi(1-e^{2i\alpha})\right)^{-M/2} \int_{\mathbb{R}^{m|2n}} \exp \frac{-4e^{i\alpha}\langle x, y\rangle + (1+e^{2i\alpha})(x^2+y^2)}{2-2e^{2i\alpha}}.$$

Proof. By definition we have

$$\mathcal{F}^{a}_{m|2n} = \exp \frac{i\alpha}{2}(\Delta - x^2 - M).$$

We can evaluate this exponential as

$$\begin{aligned}
\mathcal{F}^{a}_{m|2n} &= \exp \frac{i\alpha}{2}(\Delta_b - \underline{x}^2 - m) \exp \frac{i\alpha}{2}(\Delta_f - \underline{x\grave{}}^2 + 2n)\\
&= \mathcal{F}^{a}_{m|0} \exp i\alpha(2\sum_j \partial_{x\grave{}_{2j-1}}\partial_{x\grave{}_{2j}} - \frac{1}{2}\sum_j x\grave{}_{2j-1}x\grave{}_{2j} + n)\\
&= \mathcal{F}^{a}_{m|0} \prod_j \exp i\alpha(2\partial_{x\grave{}_{2j-1}}\partial_{x\grave{}_{2j}} - \frac{1}{2}x\grave{}_{2j-1}x\grave{}_{2j} + 1).
\end{aligned}$$

The integral representation of the bosonic fractional Fourier transform is given by (see [77])

$$\mathcal{F}^{a}_{m|0} = \left(\pi(1-e^{2i\alpha})\right)^{-m/2} \int_{\mathbb{R}^{m}} dV(\underline{x}) \exp \frac{-4e^{i\alpha}\langle \underline{x}, \underline{y}\rangle + (1+e^{2i\alpha})(\underline{x}^2+\underline{y}^2)}{2-2e^{2i\alpha}}.$$

Combining this formula with lemma 21 gives the desired result. □

Remark 26. *Note again that the kernel of the fractional Fourier transform is symmetric: $K(x,y) = K(y,x)$. Moreover, it would be impossible to introduce a fractional power of the Fourier transform in fermionic space using an orthogonal kernel. Indeed, if so we would not dispose of a corresponding Laplace operator nor of a generalization of the norm squared of a vector since $x\grave{}_i^2 = 0 = \partial^2_{x\grave{}_i}$.*

Remark 27. *Note that we could also study a slightly more general transform, where we choose α independently for each commuting variable and each pair of anti-commuting variables.*

In the following lemma we give the basic calculus properties of the fractional Fourier transform.

Lemma 22. *If $g \in \mathcal{S}(\mathbb{R}^m)_{m|2n}$, then the following relations hold:*

$$
\begin{aligned}
\mathcal{F}^{a}_{m|2n}(\partial_{x_i} g) &= [\cos\alpha\partial_{y_i} - i\sin\alpha y_i]\mathcal{F}^{a}_{m|2n}(g)\\
\mathcal{F}^{a}_{m|2n}(\partial_{\grave{x}_{2i}} g) &= [\cos\alpha\partial_{\grave{y}_{2i}} - \tfrac{i}{2}\sin\alpha\grave{y}_{2i-1}]\mathcal{F}^{a}_{m|2n}(g)\\
\mathcal{F}^{a}_{m|2n}(\partial_{\grave{x}_{2i-1}} g) &= [\cos\alpha\partial_{\grave{y}_{2i-1}} + \tfrac{i}{2}\sin\alpha\grave{y}_{2i}]\mathcal{F}^{a}_{m|2n}(g)\\
\mathcal{F}^{a}_{m|2n}(x_i g) &= [-i\sin\alpha\partial_{y_i} + \cos\alpha y_i]\mathcal{F}^{a}_{m|2n}(g)\\
\mathcal{F}^{a}_{m|2n}(\grave{x}_{2i} g) &= [2i\sin\alpha\partial_{\grave{y}_{2i-1}} + \cos\alpha\grave{y}_{2i}]\mathcal{F}^{a}_{m|2n}(g)\\
\mathcal{F}^{a}_{m|2n}(\grave{x}_{2i-1} g) &= [-2i\sin\alpha\partial_{\grave{y}_{2i}} + \cos\alpha\grave{y}_{2i-1}]\mathcal{F}^{a}_{m|2n}(g).
\end{aligned}
$$

As a consequence one has that

$$\mathcal{F}^{a}_{m|2n}\left((\partial_x + x)g\right) = e^{i\alpha}(\partial_x + x)\mathcal{F}^{a}_{m|2n}\left(g\right).$$

Proof. The formulae for x_i and ∂_{x_i} can be found in [77]. The formulae for $\grave{x}_i$ and $\partial_{\grave{x}_i}$ follow by explicitly computing the action of $\mathcal{F}^{a}_{0|2}$ on $\grave{x}_i g$ and on $\partial_{\grave{x}_i} g$ with $g \in \Lambda_2$.
Note that in the case where $\alpha = \pm\pi/2$ we reobtain lemma 18. □

7.5 The Radon transform in superspace

The classical Radon transform in $\mathbb{R}^m$ is defined by

$$\mathcal{R}_{m|0}(f)(\underline{\omega}, p) = \int_{\mathbb{R}^m} \delta(\underline{x}\cdot\underline{\omega} - p) f(\underline{x}) dV(\underline{x}).$$

It is a transform which maps a function defined on $\mathbb{R}^m$ to the integral of this function over all hyperplanes in $\mathbb{R}^m$, i.e. to a function defined on the half cylinder $\mathbb{S}^{m-1} \times \mathbb{R}^+$ with co-ordinates $(\underline{\omega}, p)$. For the mathematical theory of the Radon transform we refer the reader to e.g. [63]. A nice overview of its properties and applications can be found in [36].

In this section we will use the connection between the Radon transform and the Fourier transform for its definition in superspace. This connection is expressed by the following theorem (see [63]).

Theorem 52 (central-slice)**.** *One has that*

$$\mathcal{R}_{m|0}(f)(\underline{\omega}, p) = (2\pi)^{\frac{m}{2}-1} \int_{-\infty}^{\infty} e^{ipr} \mathcal{F}^{-}_{m|0}(f)(r\underline{\omega}) dr.$$

This theorem says that the Radon transform of a function f is obtained by first taking the Fourier transform of the function, followed by a second Fourier transform with respect to the radius r. As we have developed in the foregoing sections a general theory of Fourier transforms in superspace, we are able to use this relation as a definition of a super Radon transform.

Definition 14. *The super Radon transform of a function $f \in \mathcal{S}(\mathbb{R}^m)_{m|2n}$ is defined by*

$$\mathcal{R}_{m|2n}(f)(\omega, p) = (2\pi)^{\frac{M}{2}-1} \int_{-\infty}^{\infty} e^{ipr} \left[\mathcal{F}^{-}_{m|2n}(f)(r\omega) \mod \omega^2 + 1 \right] dr.$$

In this definition $\omega^2 = \sum_{j=1}^{n} \grave{\omega}_{2j-1}\grave{\omega}_{2j} - \sum_{j=1}^{m} \omega_j^2 = -1$ is the algebraic relation defining the supersphere. Also note that the transform is only well defined if $m \neq 0$, meaning that there is no purely fermionic Radon transform.

This Radon transform has the following basic properties.

Lemma 23. *If $f \in \mathcal{S}(\mathbb{R}^m)_{m|2n}$, then one has*

$$\begin{array}{rcl} \mathcal{R}_{m|2n}(\partial_{x_i} g) & = & \omega_i \frac{\partial}{\partial p} \mathcal{R}_{m|2n}(g) \\ \mathcal{R}_{m|2n}(\partial_{\grave{x}_{2i}} g) & = & \frac{1}{2} \grave{\omega}_{2i-1} \frac{\partial}{\partial p} \mathcal{R}_{m|2n}(g) \\ \mathcal{R}_{m|2n}(\partial_{\grave{x}_{2i-1}} g) & = & -\frac{1}{2} \grave{\omega}_{2i} \frac{\partial}{\partial p} \mathcal{R}_{m|2n}(g). \end{array}$$

Proof. Immediate, using lemma 18. □

Now we compute the generalized Radon transform of a well-chosen basis $\widetilde{\psi_{j,k,l}}(x)$ of $\mathcal{S}(\mathbb{R}^m) \otimes \Lambda_{2n}$. If we put

$$\begin{array}{rcl} \widetilde{\psi_{j,k,l}}(x) & = & (\partial_x)^{2j} H_k^{(l)} \exp x^2/2 \\ & = & \widetilde{CH}_{2j,M,k}(x) H_k^{(l)} \exp x^2/2, \end{array}$$

we can calculate that

$$\begin{array}{rcl} \mathcal{R}_{m|2n}(\widetilde{\psi_{j,k,l}}) & = & (-\omega)^{2j} (\frac{\partial}{\partial p})^{2j} \mathcal{R}_{m|2n}(H_k^{(l)} \exp x^2/2) \\ & = & (-1)^j (2\pi)^{\frac{M}{2}-1} (\frac{\partial}{\partial p})^{2j} \int_{-\infty}^{\infty} e^{ipr} \mathcal{F}^{-}_{m|2n}(H_k^{(l)} \exp x^2/2)(r\omega) dr \\ & = & (-1)^j (2\pi)^{\frac{M}{2}-1} (-i)^k (\frac{\partial}{\partial p})^{2j} \int_{-\infty}^{\infty} e^{ipr} H_k^{(l)}(\omega) r^k \exp -r^2/2 dr \end{array}$$

$$
\begin{aligned}
&= (-1)^j (2\pi)^{\frac{M}{2}-1} (-i)^k (\frac{\partial}{\partial p})^{2j+k} (-i)^k \int_{-\infty}^{\infty} e^{ipr} \exp -r^2/2 dr H_k^{(l)}(\omega) \\
&= (-1)^j (2\pi)^{\frac{M-1}{2}} (-1)^{2j+k} (\frac{\partial}{\partial p})^{2j+k} \exp -p^2/2 H_k^{(l)}(\omega) \\
&= (-1)^j (2\pi)^{\frac{M-1}{2}} \widetilde{H}_{2j+k}(p) \exp -p^2/2 H_k^{(l)}(\omega)
\end{aligned}
$$

with $\widetilde{H}_{2j+k}(p)$ the classical Hermite polynomial of degree $2j+k$.

Hence we may conclude that the Radon transform maps the basis $\widetilde{\psi_{j,k,l}}(x)$ of $\mathcal{S}(\mathbb{R}^m) \otimes \Lambda_{2n}$ into $\mathcal{S}(\mathbb{R}) \otimes (\oplus_{i=0}^{\infty} \mathcal{H}_i)$.

Chapter 8

A Cauchy integral formula in superspace

One of the most interesting features of Clifford analysis is that it allows for the construction of several nice Cauchy-type integral formulae in higher dimensions (see e.g. [88, 95, 89]). The aim of the present chapter is hence to show that one can also obtain a generalization to superspace of Cauchy's integral formula. This is an important result, because it is more or less equivalent to searching for a Stokes' formula in superspace, connecting integration over a supermanifold with integration over its boundary.

Note that there are versions of Stokes' formula known for supermanifolds, see e.g. the work of Palamodov in [78]. In that approach however, rather complicated machinery of algebraic geometry is used. Our aim is to use instead the framework of hypercomplex analysis developed in the previous chapters to obtain a Cauchy formula in a more straightforward way.

The advantage of our framework is also that it will allow us to predict the form of the desired formula by analogy with the classical case. Moreover we will obtain that the boundary of a supermanifold consists of two parts, which can be interpreted as the even and the odd boundary.

As a consequence of this integral formula, we will be able to construct a Cauchy-Pompeiu formula and a Cauchy representation formula for monogenic functions in superspace, although not all nice properties from the complex plane will be preserved (such as e.g. Morera's theorem).

The chapter is organized as follows. We start with a technical result needed in the sequel. Then we briefly discuss the classical Cauchy formula in $\mathbb{R}^m$. This will provide us with the necessary ideas to construct a Cauchy formula in purely fermionic space. In the next section we consider the general case, which necessitates the construction of an appropriate surface-element and a corresponding volume-element. Finally, a few corollaries to Cauchy's integral formula are discussed.

The results of this chapter have been submitted for publication, see [27].

8.1 An auxiliary result

Recall that if $R_{2t} \in \mathcal{P}_{2t}$, then the following holds (see lemma 2):

$$\Delta^{t+1}(x^2 R_{2t}) = 4(t+1)(M/2+t)\Delta^t(R_{2t}).$$

From this formula we now derive the following important result. Let $m = 0$ and let $R_{2n-2k} \in \mathcal{P}_{2n-2k}$. Then

$$\begin{aligned}
\partial_{\underline{\grave{x}}}^{2n}(\underline{\grave{x}}^{2k} R_{2n-2k}) &= 4n(-n+n-1)\partial_{\underline{\grave{x}}}^{2n-2}(\underline{\grave{x}}^{2k-2} R_{2n-2k}) \\
&= 4n(-1)4(n-1)(-n+n-2)\partial_{\underline{\grave{x}}}^{2n-4}(\underline{\grave{x}}^{2k-4} R_{2n-2k}) \\
&= 4^2 n(n-1)(-1)(-2)\partial_{\underline{\grave{x}}}^{2n-4}(\underline{\grave{x}}^{2k-4} R_{2n-2k}) \\
&= \ldots \\
&= 4^k n(n-1)\ldots(n-k+1)(-1)(-2)\ldots(-k) \\
&\quad \times \partial_{\underline{\grave{x}}}^{2n-2k}(R_{2n-2k}) \\
&= (-1)^k 4^k \frac{n!k!}{(n-k)!}\partial_{\underline{\grave{x}}}^{2n-2k}(R_{2n-2k}).
\end{aligned}$$

We define for further use the numerical coefficient

$$c(n,k) = (-1)^k 4^k \frac{n!k!}{(n-k)!}.$$

8.2 The limit cases

In this section we discuss the Cauchy theorem in the two limit cases, namely the purely bosonic case and the purely fermionic case. We start with the bosonic case.

8.2.1 The bosonic Cauchy theorem

Let Ω be an open set in $\mathbb{R}^m$, Σ a compact oriented differentiable m-dimensional manifold in Ω and $\partial\Sigma$ its smooth boundary. Then by introducing the following vector-valued surface-element

$$d\sigma_{\underline{x}} = \sum_{j=1}^{m} (-1)^{j+1} e_j \widehat{dx_j}, \qquad \widehat{dx_j} = dx_1 \dots dx_{j-1} dx_{j+1} \dots dx_m$$

and the volume-element

$$dV(\underline{x}) = dx_1 \dots dx_m,$$

where the exterior product of differential forms is understood, we have the following (classical) theorem (see e.g. [13]).

Theorem 53 (Bosonic Cauchy theorem)**.** *Let f and g be C^1-functions defined on Ω with values in $\mathbb{R}_{0,m}$. Then one has*

$$\int_{\partial\Sigma} f d\sigma_{\underline{x}}\, g = \int_{\Sigma} [(f\partial_{\underline{x}})g + f(\partial_{\underline{x}} g)] dV(\underline{x}).$$

The proof follows from a direct application of Stokes' theorem, because $d(f d\sigma_{\underline{x}}\, g) = [(f\partial_{\underline{x}})g + f(\partial_{\underline{x}} g)]dV(\underline{x})$.

8.2.2 The fermionic Cauchy theorem

We want to construct a formula which looks like

$$\int_B f d\sigma_{\underline{\grave{x}}} g = \int_B [(-(f\partial_{\underline{\grave{x}}})g + f(\partial_{\underline{\grave{x}}} g))] dV(\underline{\grave{x}}) \tag{8.1}$$

where $dV(\underline{\grave{x}})$ is a suitable volume-element in purely fermionic superspace and $d\sigma_{\underline{\grave{x}}}$ a corresponding surface-element. We first note that in the bosonic case, upon introducing the vector differential $d\underline{x} = \sum_{i=1}^m e_i dx_i$, the surface- and volume-elements can be expressed as follows:

$$\begin{aligned} (d\underline{x})^m &= m!\, e_1 \dots e_m\, dV(\underline{x}) \\ (d\underline{x})^{m-1} &= -(m-1)!\, d\sigma_{\underline{x}}\, e_1 \dots e_m. \end{aligned} \tag{8.2}$$

Recall that the Berezin integral is defined by deriving with respect to all the anti-commuting variables. This means that, when integrating a function f, only the term in $\grave{x}_1 \ldots \grave{x}_{2n}$ will contribute to the integral. Hence, it would make sense to define the volume-element in fermionic superspace by

$$\grave{x}_1 \ldots \grave{x}_{2n} = \frac{\underline{\grave{x}}^{2n}}{n!}.$$

Comparing this with formula (8.2) we conclude that a good candidate for $d\sigma_{\underline{\grave{x}}}$ would then be $\underline{\grave{x}}^{2n-1}/(n-1)!$.

Although this approach would indeed yield a Cauchy type formula in superspace, it is in fact a rather meager result. Indeed, if we expand f and g into homogeneous components

$$\begin{array}{rcll} f & = & f_0 + f_1 + \ldots f_{2n}, & \quad f_i \in \mathcal{P}_i \otimes \mathcal{C} \\ g & = & g_0 + g_1 + \ldots g_{2n}, & \quad g_i \in \mathcal{P}_i \otimes \mathcal{C}, \end{array}$$

we obtain

$$f\underline{\grave{x}}^{2n-1} g = f_0 \underline{\grave{x}}^{2n-1} g_1 + f_1 \underline{\grave{x}}^{2n-1} g_0,$$

because there are no polynomials of degree higher than $2n$. In formula (8.1), only the terms f_0, f_1 and g_0, g_1 of the functions f and g would thus play a role. This can be extended by introducing the following definitions instead:

$$d\sigma_{\underline{\grave{x}}} = -2\left(\underline{\grave{x}} + \frac{\underline{\grave{x}}^3}{1!} + \frac{\underline{\grave{x}}^5}{2!} + \ldots + \frac{\underline{\grave{x}}^{2n-1}}{(n-1)!}\right) = -2\underline{\grave{x}} \exp(\underline{\grave{x}}^2)$$

and

$$dV(\underline{\grave{x}}) = \frac{\underline{\grave{x}}^2}{1!} + \frac{\underline{\grave{x}}^4}{2!} + \ldots + \frac{\underline{\grave{x}}^{2n}}{n!} = \exp(\underline{\grave{x}}^2) - 1$$

as will become clear in the sequel. In this way, more components of the functions f and g will contribute to the resulting formula (see theorem 54).

Let us start with the following technical lemma.

Lemma 24. *Suppose $f \in \mathcal{P}_i \otimes \mathcal{C}$, $g \in \mathcal{P}_j \otimes \mathcal{C}$ with $i + j + 1 = 2n - 2k$. Then one has*

$$\partial_{\underline{\grave{x}}}^{2n-2k}(f\underline{\grave{x}}g) = 2(n-k)\partial_{\underline{\grave{x}}}^{2n-2k-2}\left[-(f\partial_{\underline{\grave{x}}})g + f(\partial_{\underline{\grave{x}}}g)\right].$$

Proof. The proof is done by induction on k. We first consider the case where $k = n - 1$. We then need to prove that

$$\begin{array}{rcl} \partial_{\underline{\grave{x}}}^2(f_1\underline{\grave{x}}g_0) & = & -2(f_1\partial_{\underline{\grave{x}}})g_0 \\ \partial_{\underline{\grave{x}}}^2(f_0\underline{\grave{x}}g_1) & = & 2f_0(\partial_{\underline{\grave{x}}}g_1) \end{array}$$

with $f_i, g_i \in \mathcal{P}_i \otimes \mathcal{C}$. We give the proof for the first equation:

$$\begin{aligned}
\partial_{\underline{\grave{x}}}^2(f_1\underline{\grave{x}}g_0) &= 4\sum_{j=0}^{n}\partial_{\grave{x}_{2i-1}}\partial_{\grave{x}_{2i}}(f_1\underline{\grave{x}})g_0\\
&= 4\sum_{j=0}^{n}\partial_{\grave{x}_{2i-1}}((\partial_{\grave{x}_{2i}}f_1)\underline{\grave{x}} - f_1\grave{e}_{2i})g_0\\
&= 4\sum_{j=0}^{n}((\partial_{\grave{x}_{2i}}f_1)\grave{e}_{2i-1} - (\partial_{\grave{x}_{2i-1}}f_1)\grave{e}_{2i})g_0\\
&= 4\sum_{j=0}^{n}((f_1\partial_{\grave{x}_{2i}})\grave{e}_{2i-1} - (f_1\partial_{\grave{x}_{2i-1}})\grave{e}_{2i})g_0\\
&= -2(f_1\partial_{\underline{\grave{x}}})g_0.
\end{aligned}$$

The other expression is obtained in a similar way. We proceed by induction. Suppose the lemma is proven for $k = l, \ldots, n-1$, then we prove that it also holds for $k = l-1$. This means that we have to prove that

$$\partial_{\underline{\grave{x}}}^{2n-2l+2}(f\underline{\grave{x}}g) = 2(n-l+1)\partial_{\underline{\grave{x}}}^{2n-2l}\left[-(f\partial_{\underline{\grave{x}}})g + f(\partial_{\underline{\grave{x}}}g)\right]$$

where $f \in \mathcal{P}_i \otimes \mathcal{C}$, $g \in \mathcal{P}_j \otimes \mathcal{C}$ with $i+j+1 = 2n-2l+2$. We do the calculation in the case where i is odd, the other case being similar. Then

$$\begin{aligned}
&\partial_{\underline{\grave{x}}}^{2n-2l+2}(f\underline{\grave{x}}g)\\
= \;& 4\partial_{\underline{\grave{x}}}^{2n-2k}\sum_{i=1}^{n}\partial_{\grave{x}_{2i-1}}\partial_{\grave{x}_{2i}}(f\underline{\grave{x}}g)\\
= \;& 4\partial_{\underline{\grave{x}}}^{2n-2l}\sum_{i=1}^{n}\partial_{\grave{x}_{2i-1}}\left((\partial_{\grave{x}_{2i}}f)\underline{\grave{x}}g - f\grave{e}_{2i}g + f\underline{\grave{x}}(\partial_{\grave{x}_{2i}}g)\right)\\
= \;& 4\partial_{\underline{\grave{x}}}^{2n-2l}\sum_{i=1}^{n}\big((\partial_{\grave{x}_{2i-1}}\partial_{\grave{x}_{2i}}f)\underline{\grave{x}}g + (\partial_{\grave{x}_{2i}}f)\grave{e}_{2i-1}g - (\partial_{\grave{x}_{2i}}f)\underline{\grave{x}}(\partial_{\grave{x}_{2i-1}}g)\\
&-(\partial_{\grave{x}_{2i-1}}f)\grave{e}_{2i}g + f\grave{e}_{2i}(\partial_{\grave{x}_{2i-1}}g) + (\partial_{\grave{x}_{2i-1}}f)\underline{\grave{x}}(\partial_{\grave{x}_{2i}}g)\\
&-f\grave{e}_{2i-1}(\partial_{\grave{x}_{2i}}g) + f\underline{\grave{x}}(\partial_{\grave{x}_{2i-1}}\partial_{\grave{x}_{2i}}g)\big)
\end{aligned}$$

$$
\begin{aligned}
&= 4\partial_{\underline{\grave{x}}}^{2n-2l}\sum_{i=1}^{n}\big((\partial_{\grave{x}_{2i-1}}\partial_{\grave{x}_{2i}}f)\underline{\grave{x}}g + (f\partial_{\grave{x}_{2i}})\grave{e}_{2i-1}g - (\partial_{\grave{x}_{2i}}f)\underline{\grave{x}}(\partial_{\grave{x}_{2i-1}}g)\\
&\quad -(f\partial_{\grave{x}_{2i-1}})\grave{e}_{2i}g + f\grave{e}_{2i}(\partial_{\grave{x}_{2i-1}}g) + (\partial_{\grave{x}_{2i-1}}f)\underline{\grave{x}}(\partial_{\grave{x}_{2i}}g)\\
&\quad -f\grave{e}_{2i-1}(\partial_{\grave{x}_{2i}}g) + f\underline{\grave{x}}(\partial_{\grave{x}_{2i-1}}\partial_{\grave{x}_{2i}}g)\big)\\
&= \partial_{\underline{\grave{x}}}^{2n-2l}\left[-2(f\partial_{\underline{\grave{x}}})g + 2f(\partial_{\underline{\grave{x}}}g)\right]\\
&\quad +\partial_{\underline{\grave{x}}}^{2n-2l}\Bigg((\partial_{\underline{\grave{x}}}^2 f)\underline{\grave{x}}g + f\underline{\grave{x}}(\partial_{\underline{\grave{x}}}^2 g) + 4\sum_i(\partial_{\grave{x}_{2i-1}}f)\underline{\grave{x}}(\partial_{\grave{x}_{2i}}g)\\
&\quad -4\sum_i(\partial_{\grave{x}_{2i}}f)\underline{\grave{x}}(\partial_{\grave{x}_{2i-1}}g)\Bigg)
\end{aligned}
$$

where we have used the fact that for $F_i \in \mathcal{P}_i$

$$
\begin{aligned}
\partial_{\grave{x}_k}F_i &= F_i\partial_{\grave{x}_k} \quad \text{if } i \text{ is odd}\\
&= -F_i\partial_{\grave{x}_k} \quad \text{if } i \text{ is even.}
\end{aligned}
$$

We can now apply the induction hypothesis to the last line of the previous calculation. This yields

$$
\begin{aligned}
&\partial_{\underline{\grave{x}}}^{2n-2l}\Bigg((\partial_{\underline{\grave{x}}}^2 f)\underline{\grave{x}}g + f\underline{\grave{x}}(\partial_{\underline{\grave{x}}}^2 g) + 4\sum_i(\partial_{\grave{x}_{2i-1}}f)\underline{\grave{x}}(\partial_{\grave{x}_{2i}}g)\\
&\quad -4\sum_i(\partial_{\grave{x}_{2i}}f)\underline{\grave{x}}(\partial_{\grave{x}_{2i-1}}g)\Bigg)\\
&= 2(n-l)\partial_{\underline{\grave{x}}}^{2n-2l-2}\Bigg(-((\partial_{\underline{\grave{x}}}^2 f)\partial_{\underline{\grave{x}}})g + (\partial_{\underline{\grave{x}}}^2 f)\partial_{\underline{\grave{x}}}g - (f\partial_{\underline{\grave{x}}})(\partial_{\underline{\grave{x}}}^2 g) + f(\partial_{\underline{\grave{x}}}^3 g)\\
&\quad +4\sum_i[-((\partial_{\grave{x}_{2i-1}}f)\partial_{\underline{\grave{x}}})(\partial_{\grave{x}_{2i}}g) + (\partial_{\grave{x}_{2i-1}}f)\partial_{\underline{\grave{x}}}(\partial_{\grave{x}_{2i}}g)]\\
&\quad +4\sum_i[((\partial_{\grave{x}_{2i}}f)\partial_{\underline{\grave{x}}})(\partial_{\grave{x}_{2i-1}}g) - (\partial_{\grave{x}_{2i}}f)\partial_{\underline{\grave{x}}}(\partial_{\grave{x}_{2i-1}}g)]\Bigg)\\
&= 2(n-l)\partial_{\underline{\grave{x}}}^{2n-2l-2}\Bigg(-(\partial_{\underline{\grave{x}}}^2(f\partial_{\underline{\grave{x}}}))g + (\partial_{\underline{\grave{x}}}^2 f)\partial_{\underline{\grave{x}}}g - (f\partial_{\underline{\grave{x}}})(\partial_{\underline{\grave{x}}}^2 g) + f(\partial_{\underline{\grave{x}}}^2\partial_{\underline{\grave{x}}}g)\\
&\quad -4\sum_i[(\partial_{\grave{x}_{2i-1}}(f\partial_{\underline{\grave{x}}}))(\partial_{\grave{x}_{2i}}g) + (\partial_{\grave{x}_{2i-1}}f)(\partial_{\grave{x}_{2i}}\partial_{\underline{\grave{x}}}g)]\\
&\quad +4\sum_i[(\partial_{\grave{x}_{2i}}(f\partial_{\underline{\grave{x}}}))(\partial_{\grave{x}_{2i-1}}g) + (\partial_{\grave{x}_{2i}}f)(\partial_{\grave{x}_{2i-1}}\partial_{\underline{\grave{x}}}g)]\Bigg)
\end{aligned}
$$

$$= \quad 2(n-l)\partial_{\grave{\underline{x}}}^{2n-2l}\left[-(f\partial_{\grave{\underline{x}}})g + f(\partial_{\grave{\underline{x}}}g)\right].$$

We conclude that

$$\begin{aligned}
\partial_{\grave{\underline{x}}}^{2n-2l+2}(f\grave{\underline{x}}g) &= 2\partial_{\grave{\underline{x}}}^{2n-2l}\left[-(f\partial_{\grave{\underline{x}}})g + f(\partial_{\grave{\underline{x}}}g)\right] \\
&\quad +2(n-l)\partial_{\grave{\underline{x}}}^{2n-2l}\left[-(f\partial_{\grave{\underline{x}}})g + f(\partial_{\grave{\underline{x}}}g)\right] \\
&= 2(n-l+1)\partial_{\grave{\underline{x}}}^{2n-2l}\left[-(f\partial_{\grave{\underline{x}}})g + f(\partial_{\grave{\underline{x}}}g)\right]
\end{aligned}$$

which completes the proof of the lemma. □

Using this lemma, we now calculate

$$\begin{aligned}
&\partial_{\grave{\underline{x}}}^{2n}(f\grave{\underline{x}}^{2k+1}g) \\
&= \sum_{i+j+2k+1=2n} \partial_{\grave{\underline{x}}}^{2n}(f_i\grave{\underline{x}}^{2k+1}g_j) \\
&= \sum_{i+j+2k+1=2n} c(n,k)\partial_{\grave{\underline{x}}}^{2n-2k}(f_i\grave{\underline{x}}g_j) \\
&= \sum_{i+j+2k+1=2n} c(n,k)2(n-k)\partial_{\grave{\underline{x}}}^{2n-2k-2}\left[-(f_i\partial_{\grave{\underline{x}}})g_j + f_i(\partial_{\grave{\underline{x}}}g_j)\right] \\
&= \sum_{i+j+2k+1=2n} 2(n-k)\frac{c(n,k)}{c(n,k+1)}\partial_{\grave{\underline{x}}}^{2n}(\left[-(f_i\partial_{\grave{\underline{x}}})g_j + f_i(\partial_{\grave{\underline{x}}}g_j)\right]\grave{\underline{x}}^{2k+2}) \\
&= 2(n-k)\frac{c(n,k)}{c(n,k+1)}\partial_{\grave{\underline{x}}}^{2n}(\left[-(f\partial_{\grave{\underline{x}}})g + f(\partial_{\grave{\underline{x}}}g)\right]\grave{\underline{x}}^{2k+2})
\end{aligned}$$

where f_i and g_j are the homogeneous components of f and g.

As

$$\frac{c(n,k)}{c(n,k+1)} = -\frac{1}{4(k+1)(n-k)}$$

we conclude that

$$-2\partial_{\grave{\underline{x}}}^{2n}(f\frac{\grave{\underline{x}}^{2k+1}}{k!}g) = \partial_{\grave{\underline{x}}}^{2n}(\left[-(f\partial_{\grave{\underline{x}}})g + f(\partial_{\grave{\underline{x}}}g)\right]\frac{\grave{\underline{x}}^{2k+2}}{(k+1)!}).$$

If we combine this result with the definitions of $d\sigma_{\grave{\underline{x}}}$ and $dV(\grave{\underline{x}})$ and the definition of the Berezin integral we immediately obtain the fermionic Cauchy theorem:

Theorem 54 (Fermionic Cauchy theorem)**.** *Let f and g be elements of $\Lambda_{2n} \otimes \mathcal{C}$. Then one has*

$$\int_B f d\sigma_{\underline{\grave{x}}} g = \int_B [-(f\partial_{\underline{\grave{x}}})g + f(\partial_{\underline{\grave{x}}} g)] dV(\underline{\grave{x}}).$$

Remark 28. *Note that not all properties of integration in the complex plane are still valid in superspace. We do not have e.g. Morera's theorem in the purely fermionic case. Take e.g. $f = \underline{\grave{x}}^2 P_1$ with P_1 a spherical monogenic of degree one. Then*

$$\int_B d\sigma_{\underline{\grave{x}}} f = 0$$

but f is clearly not monogenic. This has to do with the fact that we have only one 'contour' to be considered in purely fermionic space, whereas in the complex plane one integrates over all contours.

8.3 The general Cauchy theorem

We consider an open set $\Omega \subset \mathbb{R}^m$ and a compact oriented differentiable m-dimensional manifold $\Sigma \subset \Omega$ with smooth boundary $\partial\Sigma$. In the previous section we have introduced two surface-elements $d\sigma_{\underline{x}}$ and $d\sigma_{\underline{\grave{x}}}$ and two volume-elements $dV(\underline{x})$ and $dV(\underline{\grave{x}})$. Now we want to combine these elements to obtain a suitable surface-element in superspace. First note that $d\sigma_{\underline{x}} d\sigma_{\underline{\grave{x}}}$ is not a good candidate, because this would yield an integration over a formal object of codimension two. It turns out that one has to define the surface element on Σ as

$$d\sigma_x = d\sigma_{\underline{\grave{x}}}\, dV(\underline{x}) - dV(\underline{\grave{x}}) d\sigma_{\underline{x}}.$$

Note that this is a vector in the e_i and $\grave{e}_j$:

$$d\sigma_x = \sum_{j=1}^{m} (-1)^j e_j \widehat{dx_j} dV(\underline{\grave{x}}) - 2 \sum_{j=1}^{2n} \grave{e}_j \grave{x}_j \exp(\underline{\grave{x}}^2) dV(\underline{x})$$

which one would also expect a priori by analogy with the classical case.

The definition of $d\sigma_x$ forces us to define integration of an object such as $f d\sigma_x g$ in the following way. First we introduce the notations:

$$\int_{B,\Sigma} = \int_\Sigma \int_B = \int_B \int_\Sigma$$
$$\int_{B,\partial\Sigma} = \int_{\partial\Sigma} \int_B = \int_B \int_{\partial\Sigma}$$

then we define

$$\int_{B,\Sigma,\partial\Sigma} f d\sigma_x g = \int_{B,\Sigma} f d\sigma_{\underline{\grave{x}}}\, dV(\underline{x}) g - \int_{B,\partial\Sigma} f dV(\underline{\grave{x}}) d\sigma_{\underline{x}} g \tag{8.3}$$

where we note that this includes two integrations: one over the whole manifold Σ as well as one over the boundary $\partial\Sigma$. This might seem strange, but it is in fact very natural, as the two Berezin integrations $\int_B$ are actually one over the whole fermionic space and one over the fermionic boundary. In this way, both integrations in (8.3) are formally over the odd and the even boundary of a supermanifold (in the purely bosonic case there is only an even boundary).

Similarly we have the following general volume-element defined as

$$dV(x) = dV(\underline{x})dV(\underline{\grave{x}}).$$

Now we can construct the general Cauchy theorem. The two terms in (8.3) are calculated as follows, with $f, g \in C^1(\Omega)_{m|2n}$:

$$\begin{aligned}
\int_{B,\Sigma} f d\sigma_{\underline{\grave{x}}}\, dV(\underline{x}) g &= \int_{B,\Sigma} (f d\sigma_{\underline{\grave{x}}}\, g) dV(\underline{x}) \\
&= \int_{\Sigma} \left[\int_B f d\sigma_{\underline{\grave{x}}}\, g \right] dV(\underline{x}) \\
&= \int_{\Sigma} \left[\int_B [-(f\partial_{\underline{\grave{x}}})g + f(\partial_{\underline{\grave{x}}} g)] dV(\underline{\grave{x}}) \right] dV(\underline{x})
\end{aligned}$$

and

$$\begin{aligned}
\int_{B,\partial\Sigma} f dV(\underline{\grave{x}}) d\sigma_{\underline{x}} g &= \int_{B,\partial\Sigma} (f d\sigma_{\underline{x}} g) dV(\underline{\grave{x}}) \\
&= \int_B \left[\int_{\partial\Sigma} (f d\sigma_{\underline{x}} g) \right] dV(\underline{\grave{x}}) \\
&= \int_B \left[\int_{\Sigma} [(f\partial_{\underline{x}})g + f(\partial_{\underline{x}} g)] dV(\underline{x}) \right] dV(\underline{\grave{x}}).
\end{aligned}$$

So we can calculate

$$\begin{aligned}
\int_{B,\Sigma,\partial\Sigma} f d\sigma_x g &= \int_{B,\Sigma} f d\sigma_{\underline{\grave{x}}}\, dV(\underline{x}) g - \int_{B,\partial\Sigma} f dV(\underline{\grave{x}}) d\sigma_{\underline{x}} g \\
&= \int_{\Sigma} \left[\int_B [-(f\partial_{\underline{\grave{x}}})g + f(\partial_{\underline{\grave{x}}} g)] dV(\underline{\grave{x}}) \right] dV(\underline{x}) \\
&\quad - \int_B \left[\int_{\Sigma} [(f\partial_{\underline{x}})g + f(\partial_{\underline{x}} g)] dV(\underline{x}) \right] dV(\underline{\grave{x}})
\end{aligned}$$

$$\begin{aligned} &= \int_{B,\Sigma} \left[-(f\partial_{\grave{\underline{x}}})g - (f\partial_{\underline{x}})g + f(\partial_{\grave{\underline{x}}}g) - f(\partial_{\underline{x}}g)\right] dV(x) \\ &= \int_{B,\Sigma} [(f\partial_x)g + f(\partial_x g)]\, dV(x). \end{aligned}$$

Summarizing we thus obtain the following theorem.

Theorem 55 (General Cauchy theorem)**.** *Let $\Omega \subset \mathbb{R}^m$ be an open set and $\Sigma \subset \Omega$ a compact oriented differentiable m-dimensional manifold with smooth boundary $\partial\Sigma$. Let $f, g \in C^1(\Omega)_{m|2n}$. Then one has*

$$\int_{B,\Sigma,\partial\Sigma} f d\sigma_x g = \int_{B,\Sigma} [(f\partial_x)g + f(\partial_x g)]\, dV(x).$$

8.4 Consequences and applications

In this section we will discuss some corollaries of the general Cauchy theorem obtained in the previous section (see theorem 55). First we consider the case where both f and g are monogenic functions. This leads to the following.

Corollary 9. *Let f, g be right, respectively left monogenic, i.e. $f \in \mathcal{M}(\Omega)^r_{m|2n}$, $g \in \mathcal{M}(\Omega)^l_{m|2n}$. Then one has*

$$\int_{B,\Sigma,\partial\Sigma} f d\sigma_x g = 0.$$

If we put f (resp. g) equal to the constant function 1, we obtain a generalization of the well-known Cauchy theorem in the complex plane, stating that for any holomorphic function $\int_{\mathcal{C}} f(z)dz = 0$ independently of the choice of the contour $\mathcal{C}$.

Corollary 10. *Let f, g be right, respectively left monogenic in Ω. Then for every compact oriented differentiable m-dimensional manifold $\Sigma \subset \Omega$ with smooth boundary $\partial\Sigma$ one has*

$$\begin{aligned} \int_{B,\Sigma,\partial\Sigma} f d\sigma_x &= 0 \\ \int_{B,\Sigma,\partial\Sigma} d\sigma_x g &= 0. \end{aligned}$$

In the sequel, we will need the following lemma, the proof of which is classical.

Lemma 25. *Let f be a C^1-function defined in an open set $\Omega \subset \mathbb{R}^m$ containing $\underline{y}$, let $B(\underline{y}, R)$ be a ball of radius R and center $\underline{y}$ contained in Ω. Further let $\nu_k^{m|0}$ be defined as in section 6.2. Then the following holds:*

$$\begin{aligned}
\lim_{R\to 0+} \int_{B(\underline{y},R)} \nu_k^{m|0}(\underline{x}-\underline{y}) f(\underline{x}) dV(\underline{x}) &= 0, \quad \forall k \\
\lim_{R\to 0+} \int_{\partial B(\underline{y},R)} \nu_k^{m|0}(\underline{x}-\underline{y}) d\sigma_{\underline{x}} f(\underline{x}) &= \begin{cases} 0 & \forall k > 1 \\ -f(\underline{y}) & k = 1. \end{cases}
\end{aligned}$$

Now we can formulate the following theorem.

Theorem 56 (Cauchy-Pompeiu)**.** *Let $\Omega \subset \mathbb{R}^m$ be an open set and $\Sigma \subset \Omega$ a compact oriented differentiable m-dimensional manifold with smooth boundary $\partial\Sigma$. Let $g \in C^1(\Omega)_{m|2n}$ and let $\nu_1^{m|2n}$ be the fundamental solution of the super Dirac operator. Then one has*

$$\begin{aligned}
&\int_{B,\Sigma,\partial\Sigma} \nu_1^{m|2n}(x-y) d\sigma_x g(x) - \int_{B,\Sigma} \nu_1^{m|2n}(x-y)(\partial_x g(x)) dV(x) \\
= &\begin{cases} 0 & \text{if } \underline{y} \in \Omega\backslash\Sigma \\ g(y) dV(\underline{\grave{y}}) & \text{if } \underline{y} \in \overset{\circ}{\Sigma}. \end{cases}
\end{aligned}$$

Proof. Due to linearity it suffices to prove this formula for $g = g_1(\underline{x}) g_2(\underline{\grave{x}})$ where g_1 contains only commuting variables and g_2 contains only anti-commuting variables.

The formula where $\underline{y} \in \Omega\backslash\Sigma$ follows from a direct application of theorem 55. So suppose $\underline{y} \in \overset{\circ}{\Sigma}$. Then we consider a ball $\Gamma = B(\underline{y}, R)$ contained in $\overset{\circ}{\Sigma}$ and we apply theorem 55 to $\Sigma\backslash\Gamma$. We find that

$$\begin{aligned}
&\int_{B,\Sigma\backslash\Gamma,\partial(\Sigma\backslash\Gamma)} \nu_1^{m|2n}(x-y) d\sigma_x g(x) \\
= &\int_{B,\Sigma\backslash\Gamma} [(\nu_1^{m|2n}(x-y)\partial_x) g(x) + \nu_1^{m|2n}(x-y)(\partial_x g(x))] dV(x) \\
= &\int_{B,\Sigma\backslash\Gamma} \nu_1^{m|2n}(x-y)(\partial_x g(x)) dV(x),
\end{aligned}$$

as $\nu_1^{m|2n}(x-y)$ is right monogenic for $\underline{x} \neq \underline{y}$.

If $R \to 0+$ then the right-hand side tends to

$$\int_{B,\Sigma} \nu_1^{m|2n}(x-y)(\partial_x g(x))dV(x)$$

because $\nu_1^{m|2n}(x-y)(\partial_x g(x))$ is integrable. The left-hand side is calculated as

$$\begin{aligned}
& \int_{B,\Sigma\backslash\Gamma,\partial(\Sigma\backslash\Gamma)} \nu_1^{m|2n}(x-y)d\sigma_x g(x) \\
= & \int_{B,\Sigma\backslash\Gamma} \nu_1^{m|2n}(x-y)d\sigma_{\grave{\underline{x}}}dV(\underline{x})g(x) - \int_{B,\partial(\Sigma\backslash\Gamma)} \nu_1^{m|2n}(x-y)d\sigma_{\underline{x}}dV(\grave{\underline{x}})g(x) \\
= & \int_{B,\Sigma} \nu_1^{m|2n}(x-y)d\sigma_{\grave{\underline{x}}}dV(\underline{x})g(x) - \int_{B,\partial\Sigma} \nu_1^{m|2n}(x-y)d\sigma_{\underline{x}}dV(\grave{\underline{x}})g(x) \\
& - \int_{B,\Gamma} \nu_1^{m|2n}(x-y)d\sigma_{\grave{\underline{x}}}dV(\underline{x})g(x) + \int_{B,\partial\Gamma} \nu_1^{m|2n}(x-y)d\sigma_{\underline{x}}dV(\grave{\underline{x}})g(x) \\
= & \int_{B,\Sigma,\partial\Sigma} \nu_1^{m|2n}(x-y)d\sigma_x g(x) \\
& - \int_{B,\Gamma} \nu_1^{m|2n}(x-y)d\sigma_{\grave{\underline{x}}}dV(\underline{x})g(x) + \int_{B,\partial\Gamma} \nu_1^{m|2n}(x-y)d\sigma_{\underline{x}}dV(\grave{\underline{x}})g(x).
\end{aligned}$$

Now we simplify the expression in the last line. Using lemma 25 we see that only the term

$$\pi^n \nu_1^{m|0}(\underline{x}-\underline{y})\frac{(\grave{\underline{x}}-\grave{\underline{y}})^{2n}}{n!} = \nu_1^{m|0}(\underline{x}-\underline{y})\delta(\grave{\underline{x}}-\grave{\underline{y}})$$

in $\nu_1^{m|2n}(x-y)$ will play a role. This has the following result

$$\begin{aligned}
& - \int_{B,\Gamma} \nu_1^{m|2n}(x-y)d\sigma_{\grave{\underline{x}}}dV(\underline{x})g(x) + \int_{B,\partial\Gamma} \nu_1^{m|2n}(x-y)d\sigma_{\underline{x}}dV(\grave{\underline{x}})g(x) \\
= & \int_{B,\partial\Gamma} \nu_1^{m|0}(\underline{x}-\underline{y})\delta(\grave{\underline{x}}-\grave{\underline{y}})d\sigma_{\underline{x}}dV(\grave{\underline{x}})g(x) \\
= & \int_{\partial\Gamma} \nu_1^{m|0}(\underline{x}-\underline{y})d\sigma_{\underline{x}}g_1(\underline{x}) \left[\int_B \delta(\grave{\underline{x}}-\grave{\underline{y}})dV(\grave{\underline{x}})g_2(\grave{\underline{x}})\right] \\
= & \int_{\partial\Gamma} \nu_1^{m|0}(\underline{x}-\underline{y})d\sigma_{\underline{x}}g_1(\underline{x})dV(\grave{\underline{y}})g_2(\grave{\underline{y}}) \\
= & -g_1(\underline{y})dV(\grave{\underline{y}})g_2(\grave{\underline{y}}) \\
= & -g(y)dV(\grave{\underline{y}}),
\end{aligned}$$

when taking the limit $R \rightarrow 0+$ and where we have applied lemma 25 in the penultimate line. Putting all terms together completes the proof. □

Remark 29. *The result of theorem 56 is perhaps not completely as desired: we have found that the right-hand side equals $g(y)dV(\grave{\underline{y}})$. The function $dV(\grave{\underline{y}})$ is absent in the classical result (see e.g. [13]). One could propose to divide both sides of the Cauchy-Pompeiu formula by $dV(\grave{\underline{y}})$ to improve the result. This is however not possible, because $dV(\grave{\underline{y}})$ is nilpotent.*

If moreover g is left monogenic, the Cauchy-Pompeiu theorem reduces to the following:

Corollary 11. *If $g \in \mathcal{M}(\Omega)^l_{m|2n}$, then one has*

$$\int_{B,\Sigma,\partial\Sigma} \nu_1^{m|2n}(x-y)d\sigma_x g(x) \quad = \quad \begin{cases} 0 & \text{if } \underline{y} \in \Omega\backslash\Sigma \\ g(y)dV(\grave{\underline{y}}) & \text{if } \underline{y} \in \mathring{\Sigma}. \end{cases}$$

It is easy to generalize this corollary to k-monogenic functions, i.e. null-solutions of ∂_x^k. We obtain the following theorem. Note that similar formulae also exist in classical Clifford analysis, see e.g. [88] or [89] for the case of polynomial type Dirac operators.

Theorem 57. *Let $\Omega \subset \mathbb{R}^m$ be an open set and $\Sigma \subset \Omega$ a compact oriented differentiable m-dimensional manifold with smooth boundary $\partial\Sigma$. Let $g \in C^k(\Omega)_{m|2n}$ be k-monogenic, i.e. $\partial_x^k g = 0$. Then one has*

$$\int_{B,\Sigma,\partial\Sigma} \sum_{j=1}^{k} (-1)^{j+1} \nu_j^{m|2n}(x-y)d\sigma_x \partial_x^{j-1} g(x) \quad = \quad \begin{cases} 0 & \text{if } \underline{y} \in \Omega\backslash\Sigma \\ g(y)dV(\grave{\underline{y}}) & \text{if } \underline{y} \in \mathring{\Sigma}. \end{cases}$$

Proof. Similar to the proof of theorem 56, using $\partial_x \nu_j^{m|2n} = \nu_{j-1}^{m|2n}$. □

Chapter 9

Conclusions and further research

Railway termini are our gates to the glorious and the unknown. Through them we pass out into adventure and sunshine, to them, alas! we return.
E.M. Forster, Howards End.

In this thesis we have developed a theory of harmonic and Clifford analysis in superspace. We have first introduced the basic framework and the basic operators. Then we have studied polynomial null-solutions of the Laplace and Dirac operator. This allowed us to give a very detailed decomposition of the polynomial algebra generated by the commuting and anti-commuting variables. Using this decomposition we were then able to construct in a unique way an integral over the supersphere, satisfying a set of interesting properties. We then extended this integral to the whole superspace and proved that it is equivalent with the Berezin integral.

We have also studied special functions in superspace. The most important ones are the Clifford-Hermite functions, as we were able to prove that they diagonalize the generalized Fourier transform, allowing us to refine this transform to the so-called fractional Fourier transform.

One of the most important results in this thesis is certainly the Cauchy integral formula that we obtained in the last chapter. We first treated the purely fermionic case, where we could exploit our experience built in the previous chapters to obtain a formula that is formally equivalent with the well-known Cauchy formula in Clifford analysis. By defining proper surface- and volume-elements we were then able to prove a general Cauchy formula in superspace. Combining this formula with the fundamental solution of the Dirac operator yielded a Cauchy-Pompeiu representation formula for monogenic functions.

In this final chapter we wish to discuss some questions and problems that we would like to solve in the future.

9.1 Integrability conditions and syzygies

Suppose we would like to solve the equation

$$\Delta g = h$$

with g unknown. In the case where $m \neq 0$ we know that the Laplace operator is surjective and a solution can e.g. be obtained using the fundamental solution or the Fischer decomposition. In the purely fermionic case ($m = 0$) however, the Laplace operator is no longer surjective and it would be interesting to know under what conditions the equation

$$\Delta_f g = h$$

can be solved, expressed as a set of equations that h needs to satisfy. These are the so-called syzygies of the system. As a natural extension of this question, we could ask for integrability conditions for systems of the type

$$\begin{cases} \Delta_1 g &= h_1 \\ \Delta_2 g &= h_2 \\ &\vdots \\ \Delta_k g &= h_k \end{cases}$$

with g unknown and where now either the full or the fermionic Laplace operator is considered in k different sets of variables. This question is in principle treatable, because we can reformulate the problem as a large matrix with entries that are either real numbers or the bosonic Laplace operator Δ_b, acting on some vectorspace. Such problems can then be treated using the theory of algebraic analysis and Gröbner bases. Alternatively, in the case of k fermionic Laplace operators, we can consider the ideal generated by these operators in the Grassmann algebra generated by all the partial derivatives and again calculate, using Gröbner bases, the annihilator of this ideal and thus obtain the syzygies.

A more complicated question is to construct compatibility conditions for systems of the type

$$\begin{cases} \partial_{x,1} g &=& h_1 \\ \partial_{x,2} g &=& h_2 \\ & \vdots & \\ \partial_{x,k} g &=& h_k \end{cases}$$

where now the super Dirac operator is considered. On the one hand, even the treatment of a system of bosonic Dirac operators is far from trivial (see [22]), especially in low dimensions because of the appearance of unexpected syzygies (see [93]). On the other hand, because of the fact that the symplectic Clifford algebra is infinite-dimensional, the problem can no longer be formulated in terms of finite-dimensional matrices acting on some appropriate space. This means that the standard techniques no longer apply. A possible approach might be the one taken in [92] using the theory of megaforms.

9.2 A Mehler formula for the Fourier kernel

The classical Mehler formula is given by

$$\sum_{k=0}^{\infty} \frac{e^{ik\alpha}}{2^k k! \sqrt{\pi}} H_k(x) H_k(y) = \left(\pi(1-e^{2i\alpha})\right)^{-1/2} \exp \frac{2e^{i\alpha} xy - e^{2i\alpha}(x^2+y^2)}{1-e^{2i\alpha}}$$

with

$$H_k(x) = (-1)^k \exp(x^2) \frac{d^k}{dx^k} \exp(-x^2)$$

the Hermite polynomial of degree k. It gives an expansion of the kernel of the fractional Fourier transform in terms of Hermite polynomials. Note that there exist several generalizations of this formula, including one to the framework of Clifford analysis (see [11]).

As we have obtained a fractional Fourier transform in superspace and also a generalization of the Hermite polynomials, it is a natural question to ask for a Mehler formula in superspace. We were already able to obtain such a formula in the purely fermionic case, but there are still some gaps to fill in to obtain the general case.

Note that the formula obtained in [56] is not related.

9.3 Quantum mechanics in superspace

There exist several applications of superspaces in theoretical physics, such as superstringtheory, supergravity and the like. It is however also possible to develop a basic theory of quantum mechanics in superspace. This has already been the subject of previous research by other authors. So has the harmonic oscillator been treated in [52]. In the purely fermionic case there have been treatments of anharmonic oscillators, see [51, 39, 40]. Recently, also the hydrogen atom (or quantum Kepler problem) has been solved in superspace, see [105].

However, all these authors develop their own techniques, which are most of the time only applicable to the specific problems that they study, such as perturbation techniques, Lie superalgebra techniques etc. It is our aim to give a unified approach to these problems, using the framework built in this thesis.

9.3.1 A generalized Schrödinger equation

A generalization of the Schrödinger equation to superspace can be given by

$$\frac{1}{2}\Delta\psi(x_i;\grave{x}_j) + V(x_i;\grave{x}_j)\psi(x_i;\grave{x}_j) = E\psi(x_i;\grave{x}_j) \tag{9.1}$$

where the wave-function can be expanded as

$$\psi = \sum_{\nu=(\nu_1,\ldots,\nu_{2n})} \psi_\nu(x_i)\grave{x}_1^{\nu_1}\ldots\grave{x}_{2n}^{\nu_{2n}}$$

and where we use units $\hbar = m = 1$.

If we substitute the expression for ψ in formula (9.1), we obtain a complicated set of partial differential equations in the $\psi_\nu(x_i)$. A priori it does not seem feasible to solve such a system for general potentials. However, for suitable choices of the potential V, it is possible to obtain the solution to the problem. We give an overview:

- The harmonic potential $V = x^2/2$: see [52] and section 4.4.
- The purely fermionic Schrödinger equation ($m = 0$). See [51, 39, 40] and [25].
- The Coulomb potential $V = -(-x^2)^{\frac{1}{2}}$: see [105].
- The delta potential $V = -a\delta(x)$, with $a \in \mathbb{R}^+$: see [24].

In all cases, the energy spectrum is given by the classical spectrum with the super-dimension M substituted for the Euclidean dimension m.

9.3.2 Some open problems

The theory of Schrödinger operators in superspace is far from being complete. There are several questions which need to be solved. It would e.g. be interesting to obtain a longer list of exactly solvable potentials. In the first place this would mean to try to calculate the spectrum for the classically solvable potentials.

Next there is the question of self-adjointness. Except for the harmonic oscillator, most of the Hamiltonians listed in the previous section are not self-adjoint with respect to the canonically defined inner product. This is due to the fact that the velocity and position operators $\partial_{\grave{x}_i}$ and $\grave{x}_i$ are not observable in the usual sense. As all operators H can be split into the sum of two self-adjoint operators by

$$H = \frac{1}{2}(H + H^+) - \frac{1}{2i}i(H - H^+),$$

it would be interesting to try calculating the spectrum of either one of these self-adjoint parts of the original operator H and to observe the changes in the resulting spectrum.

Finally, it would also be interesting to formulate other important equations of theoretical physics in our framework, such as e.g. the Rarita-Schwinger system and its generalizations.

9.4 Dunkl operators in superspace

In the previous chapters we have made several remarks comparing the theory of harmonic and Clifford analysis in superspace with the theory of the Dunkl Laplacian. It would make sense to consider to what extent the notions of this thesis still hold when replacing the Laplace operator

$$\Delta = \Delta_b + \Delta_f$$

by the operator

$$\Delta = \Delta_h + \Delta_f$$

with Δ_h the Dunkl Laplacian relative to some finite reflection group $\mathcal{G}$. This would mean breaking the $O(m) \times Sp(2n)$ symmetry to a $\mathcal{G} \times Sp(2n)$ symmetry. It is immediately clear that several of the topics we have discussed will still be valid, such as e.g. the Clifford-Hermite polynomials. On the other hand, it would be interesting to see what meaning can be given to integration invariant under this new group action, whether we would still have Pizzetti's formula etc.

A more profound question would be whether one can deform the fermionic Laplace operator to a new operator which still acts on a Grassmann algebra, but which is now invariant with respect to a (finite) subgroup of the symplectic group.

List of notations

x_i, $i = 1, \ldots, m$	commuting (bosonic) variables
$\mathbb{R}[x_1, \ldots, x_m]$	polynomial algebra generated by the x_i
$\grave{x}_j$, $j = 1, \ldots, 2n$	anti-commuting (fermionic) variables
Λ_{2n}	Grassmann algebra generated by the $\grave{x}_j$
e_i, $i = 1, \ldots, m$	orthogonal Clifford algebra generators
$\mathbb{R}_{0,m}$	orthogonal Clifford algebra generated by the e_i with signature $(-1, \ldots, -1)$
$\grave{e}_j$, $j = 1, \ldots, 2n$	symplectic Clifford algebra generators
$\mathcal{W}_n$	symplectic Clifford algebra (or Weyl algebra) generated by the $\grave{e}_j$
$\mathcal{P} = \mathbb{R}[x_1, \ldots, x_m] \otimes \Lambda_{2n}$	polynomial algebra generated by the x_i and the $\grave{x}_j$
$\mathcal{C} = \mathbb{R}_{0,m} \otimes \mathcal{W}_n$	full Clifford algebra
$\mathcal{F}(\Omega)_{m\|2n} = \mathcal{F}(\Omega) \otimes \Lambda_{2n} \otimes \mathcal{C}$	function space in superspace associated with Ω
$\underline{x}$	bosonic vector variable
$\underline{\grave{x}}$	fermionic vector variable
$x = \underline{x} + \underline{\grave{x}}$	super vector variable
$\partial_{\underline{x}}$	bosonic Dirac operator
$\partial_{\underline{\grave{x}}}$	fermionic Dirac operator
$\partial_x = \partial_{\underline{\grave{x}}} - \partial_{\underline{x}}$	super Dirac operator
Δ_b	bosonic Laplace operator
Δ_f	fermionic Laplace operator
$\Delta = \Delta_b + \Delta_f$	super Laplace operator
$\mathbb{E}_b$	bosonic Euler operator
$\mathbb{E}_f$	fermionic Euler operator
$\mathbb{E} = \mathbb{E}_b + \mathbb{E}_f$	super Euler operator
Γ_b	bosonic Gamma operator
Γ_f	fermionic Gamma operator
Γ	super Gamma operator

Δ_{LB}	Laplace-Beltrami operator
$\mathcal{H}_k^b$	space of spherical harmonics of degree k in the commuting variables
$\mathcal{H}_k^f$	space of spherical harmonics of degree k in the anti-commuting variables
$\mathcal{H}_k$	space of spherical harmonics of degree k in both the commuting and anti-commuting variables
$\mathcal{M}_k^b$	space of spherical monogenics of degree k in the commuting variables
$\mathcal{M}_k^f$	space of spherical monogenics of degree k in the anti-commuting variables
$\mathcal{M}_k$	space of spherical monogenics of degree k in both the commuting and anti-commuting variables
$\mathbb{P}_i^k$	projection from $\mathcal{P}_k$ to $\mathcal{H}_{k-2i}$
$\mathbb{Q}_{r,s}^k$	projection from $\mathcal{H}_k$ to $f_{r,k-2r-s,s}\mathcal{H}_{k-2r-s}^b \otimes \mathcal{H}_s^f$
$\mathbb{M}_i^k$	projection from $\mathcal{P}_k \otimes \mathcal{C}$ to $x^i \mathcal{M}_{k-i}$
$H_{t,M}(M_k)(x)$	Clifford-Hermite polynomial of degree (t,k) associated with the spherical monogenic M_k
$CH_{2t,M}(H_k)(x)$	Clifford-Hermite polynomial of degree $(2t,k)$ associated with the spherical harmonic H_k
$\widetilde{H}_{t,M}(M_k)(x)$	rescaled Clifford-Hermite polynomial of degree (t,k) associated with the spherical monogenic M_k
$\widetilde{CH}_{2t,M}(H_k)(x)$	rescaled Clifford-Hermite polynomial of degree $(2t,k)$ associated with the spherical harmonic H_k
$C_{t,M}^{\alpha}(M_k)(x)$	Clifford-Gegenbauer polynomial of degree (t,k) associated with the spherical monogenic M_k
$\int_{SS}$	Pizzetti integral over the supersphere
$\mathbb{P}_{2i}^{bf}$	projection from $\mathcal{H}_{2i}$ to $f_{i,0,0}\mathcal{H}_0^b \otimes \mathcal{H}_0^f$
σ_M	area of the supersphere in M dimensions
$\int_{SB}$	Pizzetti integral over the superball
$P_n^M(t)$	Legendre polynomial of degree n in M dimensions
$\mathcal{F}_{SS}(.)$	super spherical Fourier transform
$J_\alpha(x)$	Bessel function of the first kind of order α
$\int_{\mathbb{R}^{m\|2n}}$	integral over the whole superspace
$\int_B$	Berezin integral over the anti-commuting variables
$\mathcal{M}^{l(r)}(\Omega)_{m\|2n}$	space of left (right) monogenic functions defined in Ω
$\delta(x)$	super Dirac distribution

$\nu_k^{m\vert 2n}$	fundamental solution of ∂_x^k
$\mathcal{F}^{\pm}_{m\vert 2n}(.)$	super Fourier transform
$\mathcal{F}^{a}_{m\vert 2n}(.)$	fractional Fourier transform in superspace
$\mathcal{R}_{m\vert 2n}(.)$	super Radon transform
$d\sigma_{\underline{x}}$	bosonic surface element
$d\sigma_{\underline{\grave{x}}}$	fermionic surface element
$d\sigma_{x}$	full surface element
$dV(\underline{x})$	bosonic volume element
$dV(\underline{\grave{x}})$	fermionic volume element
$dV(x)$	full volume element

Nederlandse samenvatting

Het doel van dit doctoraatsproefschrift is de ontwikkeling van een functietheorie van harmonische analyse en cliffordanalyse in zogenaamde superruimten (superspaces). Hierbij beogen we vooral een grondige studie van het integratieconcept in deze superruimten en willen we de berezinintegraal verantwoorden met behulp van technieken uit de harmonische analyse.

We geven nu een gedetailleerd overzicht van de inhoud van de verschillende hoofdstukken.

Hoofdstuk 1: Inleiding

In dit inleidend hoofdstuk geven we een korte verklaring van de onbekende trefwoorden uit de titel van dit proefschrift, namelijk cliffordanalyse en superruimten. Cliffordanalyse (zie [13, 38, 58]) enerzijds is een hogerdimensionale functietheorie van functies die waarden aannemen in een orthogonale cliffordalgebra. Het interessante van deze theorie is dat ze een verfijning vormt van harmonische analyse, omdat het kwadraat van de basisoperator, de zogenaamde diracoperator, gelijk is aan de laplace-operator. Bovendien vormt ze een veralgemening van complexe analyse tot hogere dimensies, waarbij de diracoperator de rol overneemt van de cauchy-riemannoperator. Het gevolg hiervan is dat er veralgemeningen bestaan van een aantal belangrijke resultaten uit de complexe analyse, zoals een cauchy-integraalformule.

Het concept superruimte anderzijds is afkomstig uit de theoretische fysica. Een superruimte kan worden gezien als een ruimte voorzien van twee sets van coördinaten x_i en $\grave{x}_j$, waarbij de m variabelen x_i onderling commuteren en de $2n$ variabelen $\grave{x}_j$ onderling anti-commuteren. Deze anti-commuterende variabelen kunnen worden gebruikt om fermionische vrijheidsgraden op een elegante manier te beschrijven.

Wiskundig gezien werden superruimten vooral bestudeerd vanuit twee verschillende optieken, gebruik makend van methoden uit de algebraïsche meetkunde (zie [8, 73, 69]) en de differentiaalmeetkunde (zie [47, 82]). Het doel van onderhavig werk is om deze superruimten vanuit een nieuw wiskundig perspectief te bestuderen, namelijk dat van harmonische analyse en cliffordanalyse. De gevonden resultaten werden gepubliceerd (zie [29, 32, 30, 28, 31, 23, 24]) en voor publicatie opgestuurd (zie [27, 26, 25]).

Hoofdstuk 2: Cliffordanalyse in superruimten

In dit hoofdstuk introduceren we het kader waarbinnen we cliffordanalyse wensen te ontwikkelen. In tegenstelling tot cliffordanalyse in euclidische ruimten, volstaat het niet om een orthogonale cliffordalgebra te introduceren. We moeten daarnaast ook een symplectische cliffordalgebra introduceren, die gekoppeld wordt aan de anti-commuterende variabelen. Hierna kunnen we veralgemeende vectorvariabelen definiëren, en analoog ook een diracoperator, een laplace-operator, een euleroperator en een gamma-operator. We definiëren verder de superdimensie M als de actie van de diracoperator op de veralgemeende vectorvariabele en bekomen $M = m - 2n$, het verschil van het aantal commuterende en het aantal anti-commuterende variabelen.

Vervolgens tonen we aan dat deze nieuwe operatoren een aantal commutatie- en anti-commutatierelaties gehoorzamen die volkomen analoog zijn aan de relaties waaraan de klassieke diracoperator voldoet. Verder bewijzen we dat de algebra gegenereerd door de laplace-operator, het kwadraat van de veralgemeende vectorvariabele en de euleroperator isomorf is met de lie-algebra $\mathfrak{sl}_2(\mathbb{R})$. Analoog bewijzen we dat de diracoperator en de vectorvariabele de lie-superalgebra $\mathfrak{osp}(1|2)$ genereren. Hieruit mogen we besluiten dat we effectief een representatie van harmonische analyse (resp. cliffordanalyse) geconstrueerd hebben in superruimten.

Hoofdstuk 3: Sferische harmonieken en monogenen

Dit hoofdstuk is gewijd aan de studie van polynomiale oplossingen van de super dirac- en laplace-operator. We starten met een aantal definities: een sferische harmoniek van graad k is een homogeen polynoom (in zowel de commuterende als de anti-commuterende variabelen) van graad k dat tevens een nuloplossing is van de laplace-operator. Analoog is een sferische monogeen een k-homogene polynomiale nuloplossing van de diracoperator.

Vervolgens bewijzen we de zogenaamde fischerdecompositie. Deze decompositie maakt het mogelijk om een willekeurig polynoom te schrijven als een som van sferische harmonieken vermenigvuldigd met even machten van de vectorvariabele. We illustreren dit met een grafische voorstellingswijze en we bepalen ook op twee verschillende manieren projectie-operatoren die ons de verschillende stukken in deze decompositie opleveren.

In een volgende sectie vatten we een gedetailleerde studie aan van ruimten van sferische harmonieken van graad k (die we noteren als $\mathcal{H}_k$). We beginnen met een bewijs dat de laplace-operator surjectief is op de ruimte van polynomen, wat ons in staat stelt om de dimensie van $\mathcal{H}_k$ te bepalen. Vervolgens tonen we aan dat de correcte groepsactie op de ruimte van polynomen die van $SO(m) \times Sp(2n)$ is, het product van de speciale orthogonale groep met de symplectische groep. We merken op dat $\mathcal{H}_k$ wel invariant is onder deze groepsactie, maar niet irreduciebel. We bepalen vervolgens de volledige ontbinding in irreducibele deelruimten, evenals projectie-operatoren op al deze deelruimten.

In de laatste sectie geven we een bespreking van de theorie van sferische monogenen. We starten opnieuw met de fischerdecompositie, die het nu mogelijk maakt om een polynoom te ontbinden in een som van sferische monogenen vermenigvuldigd met machten van de vectorvariabele. Dit is een verfijning van het eerder bekomen resultaat. Verder bepalen we expliciet een basis voor de ruimte van sferische monogenen van graad k met behulp van een zogenaamde cauchy-kowalewskaia-extensietechniek. Merk op dat dit slechts mogelijk is indien we over minstens één commuterende variabele beschikken.

Hoofdstuk 4: Hermite en Gegenbauer polynomen in superruimten

De Clifford-Hermite en de Clifford-Gegenbauer polynomen (zie [94, 20]) vormen hogerdimensionale veralgemeningen van de Hermite en Gegenbauer polynomen op de reële rechte. Zij hebben toepassingen onder andere in waveletanalyse (zie [14, 15, 12]).

Het doel van dit hoofdstuk is om deze twee sets van polynomen te veralgemenen tot superruimten. Hiervoor bestuderen we eerst hun definitie in $\mathbb{R}^m$. Het blijkt dat het mogelijk is om het inproduct op de gewogen L_2-ruimte zodanig te herschrijven dat er geen nood meer is aan klassieke integratie. Deze herformulering maakt het mogelijk een analoog inproduct te definiëren in de superruimte, waarbij we de euclidische dimensie vervangen door de superdimensie M. Het blijkt dat dit een goede definitie is en we zijn vervolgens in staat om zowel de Clifford-Hermite als de Clifford-Gegenbauer polynomen te definiëren in het kader van superruimten. We bewijzen verder ook een aantal interessante eigenschappen van deze polynomen, zoals recursierelaties, differentiaalvergelijking, rodriguesformule en tonen het verband aan met klassieke orthogonale polynomen op de reële rechte.

Verder introduceren we ook de zogenaamde Clifford-Hermite functies, die bestaan uit het product van een Clifford-Hermite polynoom met een gepaste gaussische functie. Het blijkt dat deze functies een basis vormen voor de ruimte van sneldalende functies.

Tenslotte geven we in dit hoofdstuk een fysische interpretatie voor de superdimensie. Het blijkt immers mogelijk te zijn de schrödingervergelijking van de harmonische oscillator te veralgemenen tot superruimten. Bovendien vormen de Clifford-Hermite functies een eigenbasis van deze vergelijking. De grondenergie van deze oscillator wordt gegeven door $M/2$, wat ook a priori te verwachten was.

Hoofdstuk 5: Integratie in superruimten

In het vijfde hoofdstuk bespreken we integratie in superruimten. De integraal die men normaal gebruikt wanneer men wil integreren over een superruimte is de zogenaamde berezinintegraal. Deze integraal wordt gewoonlijk ad hoc gedefinieerd, als een differentiatie naar alle anti-commuterende variabelen, gevolgd door een klassieke integratie van de commuterende variabelen. Het is de bedoeling van dit hoofdstuk om een wiskundige fundering voor deze integraal te construeren.

Eerst bekijken we integratie over de sfeer in $\mathbb{R}^m$. We brengen hier een oud resultaat van Pizzetti (zie [79]) in herinnering, namelijk dat de integraal over de sfeer geschreven kan worden als een oneindige som van machten van de klassieke laplace-operator, telkens in de oorsprong geëvalueerd. Deze integraal lijkt een goede kandidaat om integratie over de supersfeer te definiëren. We gaan echter eerst algemener te werk. We definiëren integratie over de supersfeer als een lineaire functionaal op de ruimte van polynomen, die enerzijds invariant is ten opzichte van de actie van $SO(m) \times Sp(2n)$ en anderzijds invariant is ten opzichte van de algebraïsche vergelijking van de supersfeer.

Het blijkt nu dat de verzameling van alle functionalen die aan deze definitie voldoen een vectorruimte van dimensie $n + 1$ vormen, wat ingezien kan worden door gebruik te maken van de fischerdecompositie en de ontbinding in irreducibele deelruimten van $\mathcal{H}_k$, bewezen in hoofdstuk 3. Als we nu verder nog eisen dat een integraal over de supersfeer ruimten van sferische harmonieken onderling orthogonaal maakt, dan blijft er nog slechts één mogelijkheid over voor integratie over de supersfeer, namelijk de klassieke formule van Pizzetti waarbij de laplace-operator en de euclidische dimensie vervangen worden door de super laplace-operator en de superdimensie.

In de volgende secties onderzoeken we een aantal interessante eigenschappen waaraan de pizzetti-integraal over de supersfeer voldoet. Zo bewijzen we veralgemeende formules van Green, die integratie over de supersfeer verbinden met integratie over de superbal. Verder tonen we een veralgemening van de stelling van Funk-Hecke aan. Deze stelling zegt dat integraaltransformaties over de supersfeer, waarbij de kern een functie is van het veralgemeende inproduct (dat nu invariant is onder $SO(m) \times Sp(2n)$), diagonaal werken op de ruimten van sferische harmonieken $\mathcal{H}_k$.

In de laatste sectie tenslotte combineren we integratie over de supersfeer met een veralgemeende notie van integratie in sferische coördinaten, waarbij de jacobiaan r^{m-1} vervangen wordt door r^{M-1}. Dit levert ons een integraal op over de volledige superruimte, die bovendien onafhankelijk is van de specifieke waarde van M. Het is dan een logische vraag om deze integraal met de berezinintegraal te vergelijken, en het blijkt dat beide gelijk zijn. In dit hoofdstuk zijn we er dus in geslaagd om de berezinintegraal in verband te brengen met het bekende begrip van integratie in euclidische ruimten.

Hoofdstuk 6: Fundamentele oplossingen

In dit hoofdstuk bepalen we eerst de fundamentele oplossing voor de super laplace- en diracoperator met behulp van een directe berekeningsmethode. Vervolgens veralgemenen we dit resultaat tot alle natuurlijk machten van de diracoperator. Hierbij dient opgemerkt te worden dat er geen fundamentele oplossing bestaat in het puur fermionische geval.

Hoofdstuk 7: Integraaltransformaties in superruimten

Het doel van dit hoofdstuk is een uitbreiding te construeren van een aantal bekende integraaltransformatie tot superruimten, namelijk de fourier-, fractionele fourier- en radontransformatie. Hierbij dient eerst opgemerkt te worden dat andere auteurs ook een fouriertransformatie in superruimten gedefinieerd hebben (zie o.a. [84, 47, 66]), maar dat daar de kern met betrekking tot de anti-commuterende variabelen invariant is onder orthogonale coördinatentransformaties. In onze opzet construeren we eerst een veralgemeend inproduct, dat invariant is onder de actie van $SO(m) \times Sp(2n)$. Aangezien dit tevens de invariantie is van de laplace-operator en van de ruimten $\mathcal{H}_k$, is dit een veelbelovende manier om integraaltransformaties in superruimten te definiëren.

Vervolgens definiëren we eerst de zogenaamde fermionische fouriertransformatie (met betrekking tot de anti-commuterende variabelen). We bestuderen de basiseigenschappen van deze transformatie, zoals calculusregels, inversie, stelling van Parseval, etc. Vervolgens tonen we aan dat sferische harmonieken vermenigvuldigd met een gepaste gaussische functie eigenfuncties zijn van deze fouriertransformaties.

Combinatie van de klassieke fouriertransformatie met de fermionische fouriertransformatie levert ons de algemene fouriertransformatie in superruimten. We onderzoeken een aantal eigenschappen van deze transformatie. Daarna bewijzen we dat de Clifford-Hermite functies de fouriertransformatie diagonaliseren. Hierbij maken we gebruik van de ontbinding van de ruimte $\mathcal{H}_k$ in irreducibele deelruimten onder de actie van $SO(m) \times Sp(2n)$. Dit laat ons vervolgens toe om een operator exponentiële gedaante op te stellen voor de fouriertransformatie. In de volgende sectie gebruiken we deze gedaante om een fractionele fouriertransformatie (zie [75, 77]) in te voeren. We bepalen tevens de kern van deze nieuwe integraaltransformatie, die een mooie veralgemening vormt van de klassieke fractionele fouriertransformatie. In de laatste sectie tenslotte definiëren we de radontransformatie in superruimten door gebruik te maken van het 'central-slice' theorema, dat ons in staat stelt de klassieke radontransformatie te schrijven als twee opeenvolgende fouriertransformaties. Opnieuw blijkt dat deze transformatie zich goed gedraagt ten opzichte van de Clifford-Hermite functies.

Hoofdstuk 8: Cauchy-integraalformule in superruimten

In dit hoofdstuk construeren we een cauchyformule in superruimten. We starten met een analyse van de klassieke cauchyformule uit cliffordanalyse, wat ons de nodige inspiratie geeft om eerst een fermionische cauchyformule op te bouwen (waarbij de integratie nu de berezinintegraal is). Deze cauchyformule heeft formeel dezelfde gedaante als de klassieke cauchyformule, maar het oppervlak- en volume-element hebben een meer ingewikkelde gedaante.

In een volgende sectie combineren we beide cauchyformules tot een algemene cauchyformule. Hierbij dient de integratie over de rand van een supervariëteit gedefinieerd te worden als de som van de integraal over de even en de oneven rand. Het uiteindelijke resultaat is opnieuw formeel equivalent met de klassieke cauchyformule.

In de laatste sectie bespreken we een aantal gevolgen van de cauchyformule in superruimten. Zo bewijzen we een cauchy-pompeiu-formule, die we vervolgens gebruiken om een representatieformule op te stellen voor monogene en k-monogene functies in superruimten.

Hoofdstuk 9: Conclusies en verder onderzoek

In het laatste hoofdstuk van dit proefschrift geven we eerst een korte samenvatting van de voorgaande hoofdstukken. Daarna bespreken we een aantal onderwerpen die we in de toekomst nog verder willen onderzoeken. Zo willen we bijvoorbeeld stelsels van laplace- of diracoperatoren bestuderen, gebruik makend van technieken uit de algebraïsche analyse. Verder willen we ook een mehlerformule bekomen, die een ontbinding van de kern van de fractionele fouriertransformatie geeft in termen van de Clifford-Hermite functies.

Ook zouden we graag een theorie van schrödingeroperatoren ontwikkelen in superruimten. Een aantal gevallen werd reeds bestudeerd, zoals de harmonische oscillator en de delta potentiaal, maar er is nood aan een meer systematische studie van analytisch oplosbare potentialen. Bovendien zijn de meeste potentialen die tot nu toe bestudeerd werden niet zelf-toegevoegd ten opzichte van het canonisch bepaalde inproduct, omdat de plaats- en snelheidsoperatoren van een anti-commuterende variabele onder toevoeging in elkaar omgezet worden.

Tenslotte zouden we ook graag nagaan in hoeverre het mogelijk is om de theorie van de zogenaamde dunkloperatoren tot superruimten te veralgemenen. Aangezien de dunkl-laplaciaan invariant is onder een eindige deelgroep $\mathcal{G}$ van de orthogonale groep, willen we met andere woorden een operator construeren die invariant is onder $\mathcal{G} \times \mathcal{H}$, met $\mathcal{H}$ een deelgroep van de symplectische groep.

Bibliography

[1] N. Aronszajn, T. M. Creese and L. J. Lipkin, *Polyharmonic functions.* Oxford Mathematical Monographs. The Clarendon Press Oxford University Press, New York, 1983. Notes taken by Eberhard Gerlach, Oxford Science Publications.

[2] S. Axler, P. Bourdon and W. Ramey, *Harmonic function theory*, second ed., vol. **137** of *Graduate Texts in Mathematics.* Springer-Verlag, New York, 2001.

[3] J. A. Baker, Integration over spheres and the divergence theorem for balls. *Amer. Math. Monthly* **104**, 1 (1997), 36–47.

[4] C. Bartocci, U. Bruzzo and D. Hernández Ruipérez, *The geometry of supermanifolds*, vol. **71** of *Mathematics and its Applications.* Kluwer Academic Publishers Group, Dordrecht, 1991.

[5] M. Batchelor, Two approaches to supermanifolds. *Trans. Amer. Math. Soc.* **258**, 1 (1980), 257–270.

[6] S. Ben Saïd and B. Ørsted, Segal-Bargmann transforms associated with finite Coxeter groups. *Math. Ann.* **334** , 2 (2006), 281–323.

[7] F. A. Berezin, *The method of second quantization.* Translated from the Russian by Nobumichi Mugibayashi and Alan Jeffrey. Pure and Applied Physics, Vol. **24**. Academic Press, New York, 1966.

[8] F. A. Berezin, *Introduction to algebra and analysis with anticommuting variables.* Moskov. Gos. Univ., Moscow, 1983. With a preface by A. A. Kirillov.

[9] F. Brackx, J. Bureš, H. De Schepper, D. Eelbode, F. Sommen and V. Souček, Fundaments of Hermitean Clifford analysis. I. Complex structure. *Complex Anal. Oper. Theory* **1**, 3 (2007), 341–365.

[10] F. Brackx, J. Bureš, H. De Schepper, D. Eelbode, F. Sommen and V. Souček, Fundaments of Hermitean Clifford analysis. II. Splitting of h-monogenic equations. *Complex Var. Elliptic Equ.* **52**, 10-11 (2007), 1063–1079.

[11] F. Brackx, N. De Schepper, K. I. Kou and F. Sommen, The Mehler formula for the generalized Clifford-Hermite polynomials. *Acta Math. Sin. (Engl. Ser.)* **23**, 4 (2007), 697–704.

[12] F. Brackx, N. De Schepper and F. Sommen, The Clifford-Gegenbauer polynomials and the associated continuous wavelet transform. *Integral Transforms Spec. Funct.* **15**, 5 (2004), 387–404.

[13] F. Brackx, R. Delanghe and F. Sommen, *Clifford analysis*, vol. **76** of *Research Notes in Mathematics.* Pitman (Advanced Publishing Program), Boston, MA, 1982.

[14] F. Brackx and F. Sommen, Clifford-Hermite wavelets in Euclidean space. *J. Fourier Anal. Appl.* **6**, 3 (2000), 299–310.

[15] F. Brackx and F. Sommen, The generalized Clifford-Hermite continuous wavelet transform. *Adv. Appl. Clifford Algebras* **11**, S1 (2001), 219–231.

[16] J. Bureš, F. Sommen, V. Souček and P. Van Lancker, Rarita-Schwinger type operators in Clifford analysis. *J. Funct. Anal.* **185**, 2 (2001), 425–455.

[17] J. Bureš, F. Sommen, V. Souček and P. Van Lancker, Symmetric analogues of Rarita-Schwinger equations. *Ann. Global Anal. Geom.* **21**, 3 (2002), 215–240.

[18] P. Cerejeiras, U. Kähler and G. Ren, Clifford analysis for finite reflection groups. *Complex Var. Elliptic Equ.* **51**, 5-6 (2006), 487–495.

[19] C. C. Chevalley, *The algebraic theory of spinors.* Columbia University Press, New York, 1954.

[20] J. Cnops, *Orthogonal functions associated with the Dirac operator.* Ph.D. thesis, Ghent university, 1989.

[21] J. Cnops, *An introduction to Dirac operators on manifolds*, vol. **24** of *Progress in Mathematical Physics*. Birkhäuser Boston Inc., Boston, MA, 2002.

[22] F. Colombo, I. Sabadini, F. Sommen and D. C. Struppa, *Analysis of Dirac systems and computational algebra*, vol. **39** of *Progress in Mathematical Physics*. Birkhäuser Boston Inc., Boston, MA, 2004.

[23] H. De Bie, Fourier transform and related integral transforms in superspace. *J. Math. Anal. Appl.* **345** (2008), 147–164.

[24] H. De Bie, Schrödinger equation with delta potential in superspace. *Phys. Lett. A* **372** (2008), 4350–4352.

[25] H. De Bie, Dimensional continuation of quantum systems. *Submitted*.

[26] H. De Bie, D. Eelbode and F. Sommen, On spherical harmonics in superspace and how they uniquely determine the Berezin integral. *Submitted*.

[27] H. De Bie and F. Sommen, A Cauchy integral formula in superspace. *Submitted*.

[28] H. De Bie and F. Sommen, A Clifford analysis approach to superspace. *Ann. Physics* **322** (2007), 2978–2993.

[29] H. De Bie and F. Sommen, Correct rules for Clifford calculus on superspace. *Adv. Appl. Clifford Algebr.* **17**, 3 (2007), 357–382.

[30] H. De Bie and F. Sommen, Hermite and Gegenbauer polynomials in superspace using Clifford analysis. *J. Phys. A: Math. Theor.* **40**, 34 (2007), 10441–10456.

[31] H. De Bie and F. Sommen, Spherical harmonics and integration in superspace. *J. Phys. A: Math. Theor.* **40**, 26 (2007), 7193–7212.

[32] H. De Bie and F. Sommen, Fischer decompositions in superspace. In *Function spaces in complex and Clifford analysis*. National University Publishers Hanoi, 2008, pp. 170–188.

[33] H. De Bie and F. Sommen, Fundamental solutions for the super Laplace and Dirac operators and all their natural powers. *J. Math. Anal. Appl.* **338** (2008), 1320–1328.

[34] M. F. E. de Jeu, The Dunkl transform. *Invent. Math.* **113**, 1 (1993), 147–162.

[35] N. De Schepper, *Multi-dimensional continuous wavelet transforms and generalized Fourier transforms in Clifford analysis.* Ph.D. thesis, Ghent university, 2006.

[36] S. R. Deans, *The Radon transform and some of its applications.* A Wiley-Interscience Publication. John Wiley & Sons Inc., New York, 1983.

[37] R. Delanghe, Clifford analysis: history and perspective. *Comput. Methods Funct. Theory* **1**, 1 (2001), 107–153.

[38] R. Delanghe, F. Sommen and V. Souček, *Clifford algebra and spinor-valued functions*, vol. **53** of *Mathematics and its Applications.* Kluwer Academic Publishers Group, Dordrecht, 1992.

[39] R. Delbourgo, L. M. Jones and M. White, Anharmonic Grassmann oscillator. *Phys. Rev. D* **40**, 8 (1989), 2716–2719.

[40] R. Delbourgo, L. M. Jones and M. White, Anharmonic Grassmann oscillator. II. *Phys. Rev. D* **41**, 2 (1990), 679–681.

[41] P. Desrosiers, L. Lapointe and P. Mathieu, Supersymmetric Calogero-Moser-Sutherland models and Jack superpolynomials. *Nuclear Phys. B* **606**, 3 (2001), 547–582.

[42] P. Desrosiers, L. Lapointe and P. Mathieu, Generalized Hermite polynomials in superspace as eigenfunctions of the supersymmetric rational CMS model. *Nuclear Phys. B* **674**, 3 (2003), 615–633.

[43] P. Desrosiers, L. Lapointe and P. Mathieu, Jack polynomials in superspace. *Comm. Math. Phys.* **242**, 1-2 (2003), 331–360.

[44] P. Desrosiers, L. Lapointe and P. Mathieu, Jack superpolynomials, superpartition ordering and determinantal formulas. *Comm. Math. Phys.* **233**, 3 (2003), 383–402.

[45] P. Desrosiers, L. Lapointe and P. Mathieu, Explicit formulae for the generalized Hermite polynomials in superspace. *J. Phys. A: Math. Gen.* **37**, 4 (2004), 1251–1268.

[46] P. Desrosiers, L. Lapointe and P. Mathieu, Classical symmetric functions in superspace. *J. Algebraic Combin.* **24**, 2 (2006), 209–238.

[47] B. DeWitt, *Supermanifolds.* Cambridge Monographs on Mathematical Physics. Cambridge University Press, Cambridge, 1984.

[48] C. F. Dunkl, Differential-difference operators associated to reflection groups. *Trans. Amer. Math. Soc.* **311**, 1 (1989), 167–183.

[49] C. F. Dunkl, Hankel transforms associated to finite reflection groups. In *Hypergeometric functions on domains of positivity, Jack polynomials, and applications (Tampa, FL, 1991)*, vol. **138** of *Contemp. Math.* Amer. Math. Soc., Providence, RI, 1992, pp. 123–138.

[50] C. F. Dunkl and Y. Xu, *Orthogonal polynomials of several variables*, vol. **81** of *Encyclopedia of Mathematics and its Applications.* Cambridge University Press, Cambridge, 2001.

[51] G. V. Dunne and I. G. Halliday, Negative-dimensional oscillators. *Nuclear Phys. B* **308**, 2-3 (1988), 589–618.

[52] R. Finkelstein and M. Villasante, Grassmann oscillator. *Phys. Rev. D* **33**, 6 (1986), 1666–1673.

[53] G. B. Folland, How to integrate a polynomial over a sphere. *Amer. Math. Monthly* **108**, 5 (2001), 446–448.

[54] L. Frappat, A. Sciarrino and P. Sorba, *Dictionary on Lie algebras and superalgebras.* Academic Press Inc., San Diego, CA, 2000. With 1 CD-ROM (Windows, Macintosh and UNIX).

[55] W. Fulton and J. Harris, *Representation theory*, vol. **129** of *Graduate Texts in Mathematics.* Springer-Verlag, New York, 1991.

[56] O. E. Galkin, Infinite-dimensional superanalogues of Mehler's formula. *Mat. Zametki* **59**, 6 (1996), 927–931.

[57] I. M. Gel′fand and G. E. Shilov, *Generalized functions. Vol. 1.* Academic Press [Harcourt Brace Jovanovich Publishers], New York, 1964 [1977]. Properties and operations, Translated from the Russian by Eugene Saletan.

[58] J. E. Gilbert and M. A. M. Murray, *Clifford algebras and Dirac operators in harmonic analysis*, vol. **26** of *Cambridge Studies in Advanced Mathematics.* Cambridge University Press, Cambridge, 1991.

[59] H. Groemer, *Geometric applications of Fourier series and spherical harmonics*, vol. **61** of *Encyclopedia of Mathematics and its Applications.* Cambridge University Press, Cambridge, 1996.

[60] K. Gürlebeck and W. Sprössig, *Quaternionic and Clifford calculus for physicists and engineers*, vol. **1** of *Mathematical Methods in Practice.* John Wiley and Sons, Chichester, 1997.

[61] K. Habermann and L. Habermann, *Introduction to symplectic Dirac operators*, vol. **1887** of *Lecture Notes in Mathematics.* Springer-Verlag, Berlin, 2006.

[62] G. J. Heckman, A remark on the Dunkl differential-difference operators. In *Harmonic analysis on reductive groups (Brunswick, ME, 1989)*, vol. **101** of *Progr. Math.* Birkhäuser Boston, Boston, MA, 1991, pp. 181–191.

[63] S. Helgason, *The Radon transform*, second ed., vol. **5** of *Progress in Mathematics.* Birkhäuser Boston Inc., Boston, MA, 1999.

[64] H. Hochstadt, *The functions of mathematical physics.* Wiley-Interscience [A division of John Wiley & Sons, Inc.], New York-London-Sydney, 1971. Pure and Applied Mathematics, Vol. XXIII.

[65] R. Howe and E.-C. Tan, *Nonabelian harmonic analysis.* Universitext. Springer-Verlag, New York, 1992. Applications of $\mathrm{SL}(2,\mathbb{R})$.

[66] A. Inoue, Foundation of real analysis on the superspace $\mathbb{R}^{m|n}$ over the ∞-dimensional Fréchet-Grassmann algebra. *J. Fac. Sci. Univ. Tokyo Sect. IA Math.* **39**, 3 (1992), 419–474.

[67] A. Inoue and Y. Maeda, Foundations of calculus on super Euclidean space $\mathbb{R}^{m|n}$ based on a Fréchet-Grassmann algebra. *Kodai Math. J.* **14**, 1 (1991), 72–112.

[68] B. Jancewicz, *Multivectors and Clifford algebra in electrodynamics.* World Scientific Publishing Co. Inc., Teaneck, NJ, 1988.

[69] B. Kostant, Graded manifolds, graded Lie theory, and prequantization. In *Differential geometrical methods in mathematical physics (Proc. Sympos., Univ. Bonn, Bonn, 1975)*. Springer, Berlin, 1977, pp. 177–306. Lecture Notes in Math., Vol. **570**.

[70] R. S. Krausshar and J. Ryan, Clifford and harmonic analysis on cylinders and tori. *Rev. Mat. Iberoamericana* **21**, 1 (2005), 87–110.

[71] R. S. Krausshar and J. Ryan, Some conformally flat spin manifolds, Dirac operators and automorphic forms. *J. Math. Anal. Appl.* **325**, 1 (2007), 359–376.

[72] V. V. Kravchenko, *Applied quaternionic analysis*, vol. **28** of *Research and Exposition in Mathematics.* Heldermann Verlag, Lemgo, 2003.

[73] D. A. Leĭtes, Introduction to the theory of supermanifolds. *Uspekhi Mat. Nauk* **35**, 1(211) (1980), 3–57, 255.

[74] P. Lounesto, *Clifford algebras and spinors*, second ed., vol. **286** of *London Mathematical Society Lecture Note Series.* Cambridge University Press, Cambridge, 2001.

[75] V. Namias, The fractional order Fourier transform and its application to quantum mechanics. *J. Inst. Math. Appl.* **25**, 3 (1980), 241–265.

[76] B. Ørsted, P. Somberg and V. Souček, The Howe duality for the Dunkl version of the Dirac operator. *Accepted for publication in Adv. Appl. Clifford Algebr.*

[77] H. Ozaktas, Z. Zalevsky and M. Kutay, *The fractional Fourier transform.* Wiley, Chichester, 2001.

[78] V. P. Palamodov, Cogitations over Berezin's integral. In *Contemporary mathematical physics*, vol. **175** of *Amer. Math. Soc. Transl. Ser. 2.* Amer. Math. Soc., Providence, RI, 1996, pp. 177–189.

[79] P. Pizzetti, Sulla media dei valori che una funzione dei punti dello spazio assume alla superficie di una sfera. *Rend. Lincei* **18** (1909), 182–185.

[80] I. R. Porteous, *Clifford algebras and the classical groups*, vol. **50** of *Cambridge Studies in Advanced Mathematics.* Cambridge University Press, Cambridge, 1995.

[81] J. M. Rabin, The Berezin integral as a contour integral. *Phys. D* **15**, 1-2 (1985), 65–70.

[82] A. Rogers, A global theory of supermanifolds. *J. Math. Phys.* **21**, 6 (1980), 1352–1365.

[83] A. Rogers, Realizing the Berezin integral as a superspace contour integral. *J. Math. Phys.* **27**, 3 (1986), 710–717.

[84] A. Rogers, Stochastic calculus in superspace. I. Supersymmetric Hamiltonians. *J. Phys. A: Math. Gen.* **25**, 2 (1992), 447–468.

[85] A. Rogers, *Supermanifolds.* World Scientific Publishing Co. Pte. Ltd., Hackensack, NJ, 2007. Theory and applications.

[86] M. Rösler, Generalized Hermite polynomials and the heat equation for Dunkl operators. *Comm. Math. Phys.* **192**, 3 (1998), 519–542.

[87] G.-C. Rota, D. Kahaner and A. Odlyzko, On the foundations of combinatorial theory. VIII. Finite operator calculus. *J. Math. Anal. Appl.* **42** (1973), 684–760.

[88] J. Ryan, Dirac operators, Schrödinger type operators in $\mathbf{C}^n$, and Huygens principle. *J. Funct. Anal.* **87**, 2 (1989), 321–347.

[89] J. Ryan, Cauchy-Green type formulae in Clifford analysis. *Trans. Amer. Math. Soc.* **347**, 4 (1995), 1331–1341.

[90] I. Sabadini and F. Sommen, Hermitian Clifford analysis and resolutions. *Math. Methods Appl. Sci.* **25**, 16-18 (2002), 1395–1413. Clifford analysis in applications.

[91] I. Sabadini and F. Sommen, Special first order systems in Clifford analysis and resolutions. *Z. Anal. Anwendungen* **21**, 1 (2002), 27–55.

[92] I. Sabadini, F. Sommen and D. C. Struppa, The Dirac complex on abstract vector variables: megaforms. *Experiment. Math.* **12**, 3 (2003), 351–364.

[93] I. Sabadini, D. C. Struppa, F. Sommen and P. Van Lancker, Complexes of Dirac operators in Clifford algebras. *Math. Z.* **239**, 2 (2002), 293–320.

[94] F. SOMMEN, Special functions in Clifford analysis and axial symmetry. *J. Math. Anal. Appl.* **130**, 1 (1988), 110–133.

[95] F. SOMMEN, Monogenic differential calculus. *Trans. Amer. Math. Soc.* **326**, 2 (1991), 613–632.

[96] F. SOMMEN, An algebra of abstract vector variables. *Portugal. Math.* **54**, 3 (1997), 287–310.

[97] F. SOMMEN, An extension of Clifford analysis towards super-symmetry. In *Clifford algebras and their applications in mathematical physics, Vol. 2 (Ixtapa, 1999)*, vol. **19** of *Progr. Phys.* Birkhäuser Boston, Boston, MA, 2000, pp. 199–224.

[98] F. SOMMEN, Clifford analysis on super-space. *Adv. Appl. Clifford Algebras* **11**, S1 (2001), 291–304.

[99] F. SOMMEN, Clifford analysis on super-space. II. In *Progress in analysis, Vol. I, II (Berlin, 2001).* World Sci. Publishing, River Edge, NJ, 2003, pp. 383–405.

[100] F. SOMMEN and N. VAN ACKER, Monogenic differential operators. *Results Math.* **22**, 3-4 (1992), 781–798.

[101] E. M. STEIN and G. WEISS, *Introduction to Fourier analysis on Euclidean spaces.* Princeton University Press, Princeton, N.J., 1971. Princeton Mathematical Series, No. 32.

[102] N. J. VILENKIN, *Special functions and the theory of group representations.* Translated from the Russian by V. N. Singh. Translations of Mathematical Monographs, Vol. **22**. American Mathematical Society, Providence, R. I., 1968.

[103] V. S. VLADIMIROV and I. V. VOLOVICH, Superanalysis. I. Differential calculus. *Teoret. Mat. Fiz.* **59**, 1 (1984), 3–27.

[104] Y. XU, Funk-Hecke formula for orthogonal polynomials on spheres and on balls. *Bull. London Math. Soc.* **32**, 4 (2000), 447–457.

[105] R. B. ZHANG, Orthosymplectic Lie superalgebras in superspace analogues of quantum Kepler problems. *Comm. Math. Phys.* **280**, (2008) 545–562.